ILMATIETEEN LAITOS
METEOROLOGISKA INSTITUTET
FINNISH METEOROLOGICAL INSTITUTE

Simulating stably stratified turbulence

Antti Hellsten

Finnish Meteorological Institute

Helsinki, Finland

Contents

- **Background and motivation**
- **Numerical methods**
- **Some theoretical aspects**
- **Some results**
- **Preliminary conclusions and outlook**

Background and motivation

By stable stratification we mean that fluid density is stratified in gravity (or other acceleration) field such that buoyancy tends to return a perturbed fluid parcel back to its equilibrium position.

Turbulent flow in stably stratified fluid is almost a rule rather than exception in atmosphere and in hydrosphere!

Density stratification is formed by natural distributions of:

- **potential temperature (and humidity) in atmosphere**
- **salinity and temperature in oceans**

Background and motivation

Atmospheric turbulence is governed not only by stratification. Other characteristic features are:

- **Earth rotation (importance increases with the scale)**
- **Anisotropic (thin) geometry**

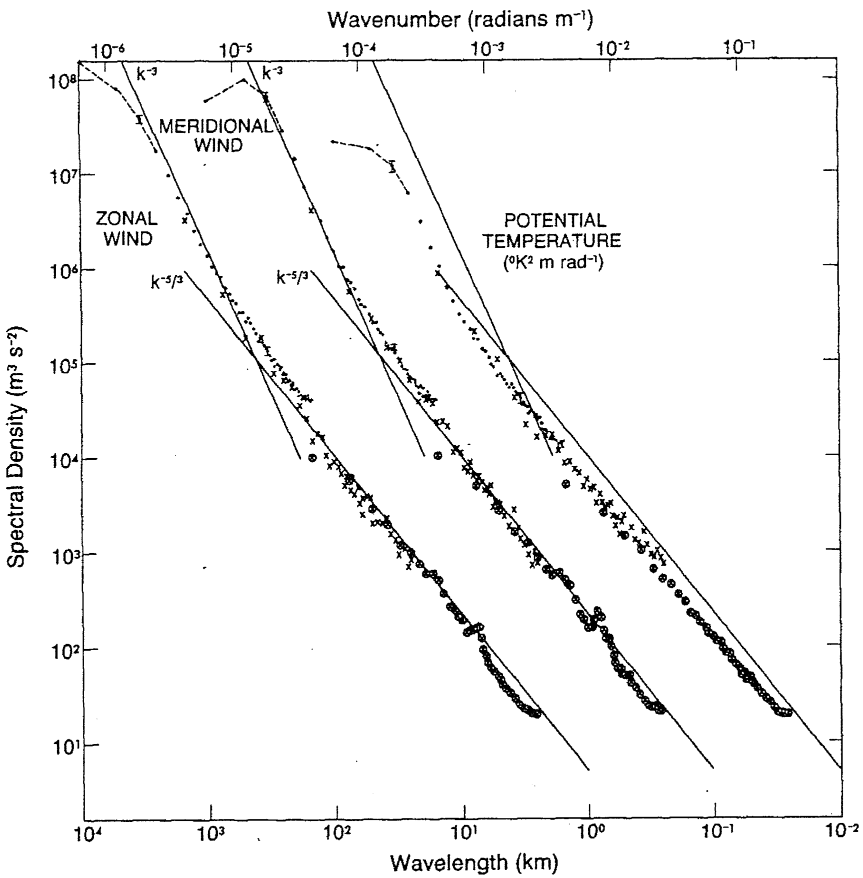
The latter feature implies that classical locally isotropic Kolmogorov turbulence is not possible at scales much larger than the largest vertical scales.

Background and motivation

Turbulent motion in earth atmosphere (excluding convective boundary layers and convective storms etc.) can be divided into three regimes based on horizontal length scales:

- **geostrophic motion at scales larger than about 3000 km strongly influenced by rotation and stratification**
- **meso-scale turbulence at scales between kilometres and hundreds of kilometres little influenced by rotation but strongly influenced by stratification rendering it very anisotropic and horizontally layered**
- **spatially intermittent Kolmogorov turbulence (overturning motion) at the smallest scales governed by local vertical shear between layers**

Background and motivation



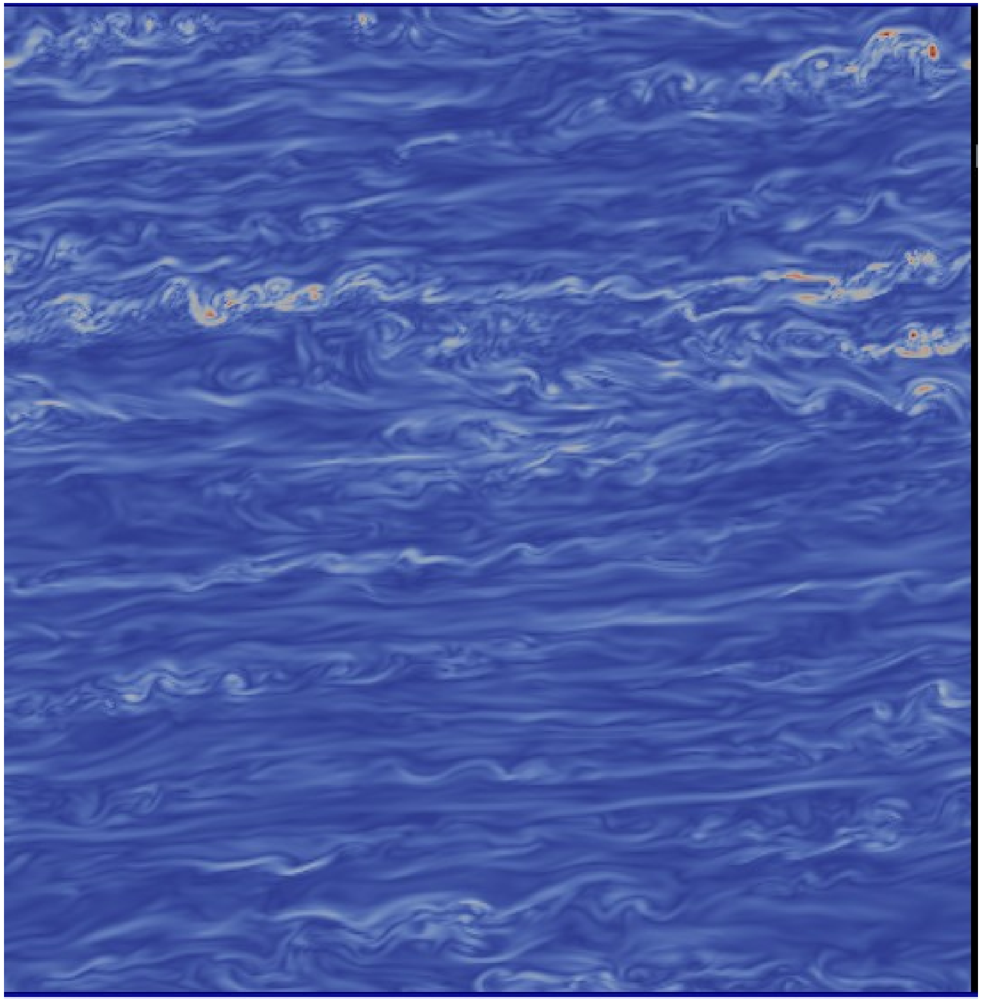
Spectra of wind and potential temperature after Nastrom & Gage (1985).

Background and motivation

An example of spatially intermittent spots of Kolmogorov turbulence in layered strongly stratified turbulence. Vorticity is shown by colours based on a DNS in a cubical domain of 384^3 nodes.

Re_T**=5200,** Fr_h**=0.02**

$R=Re_T Fr_h^2=2.5$



Background and motivation

Stably stratified turbulence has been investigated in many experimental and computational studies

- **A characteristic feature of stratified shear turbulence is that fluxes of buoyancy and momentum can be counter gradient (Gerz et al., 1989; Holt et al., 1992; …)**
- **A common observation is formation of quasi horizontal layers or "pancakes" (Herring & Metais, 1989; ...)**
- **Zigzag instability (Billant & Chomaz, 2000a, 2000b)**
- **Inviscid scaling analysis (Riley et al., 1981; Billant & Chomaz, 2001)**
- **Energy cascade (Lindborg, 2006)**
- **Transition from geostrophic to stratified turbulence (Waite & Bartello, 2006)**

Background and motivation

- **Viscous scaling analysis (Brethouwer et al., 2007)**
- **Effects of vertical mean shear on strongly stratified turbulence have not been studied**
- **Vertical mean shears typically exist also in free troposphere (not only in boundary layers)**
- **Mean shear might impose an external vertical length scale which may change the dynamics as the scaling theory (Billant & Chomaz, 2001; Brethouwer et al., 2007) assumes no imposed vertical length scales**
- **The effect of vertical mean shear is studied in the present work based on the assumption of homogeneous turbulence**

Background and motivation

Why is it important to understand the dynamics of stably stratified shear turbulence?

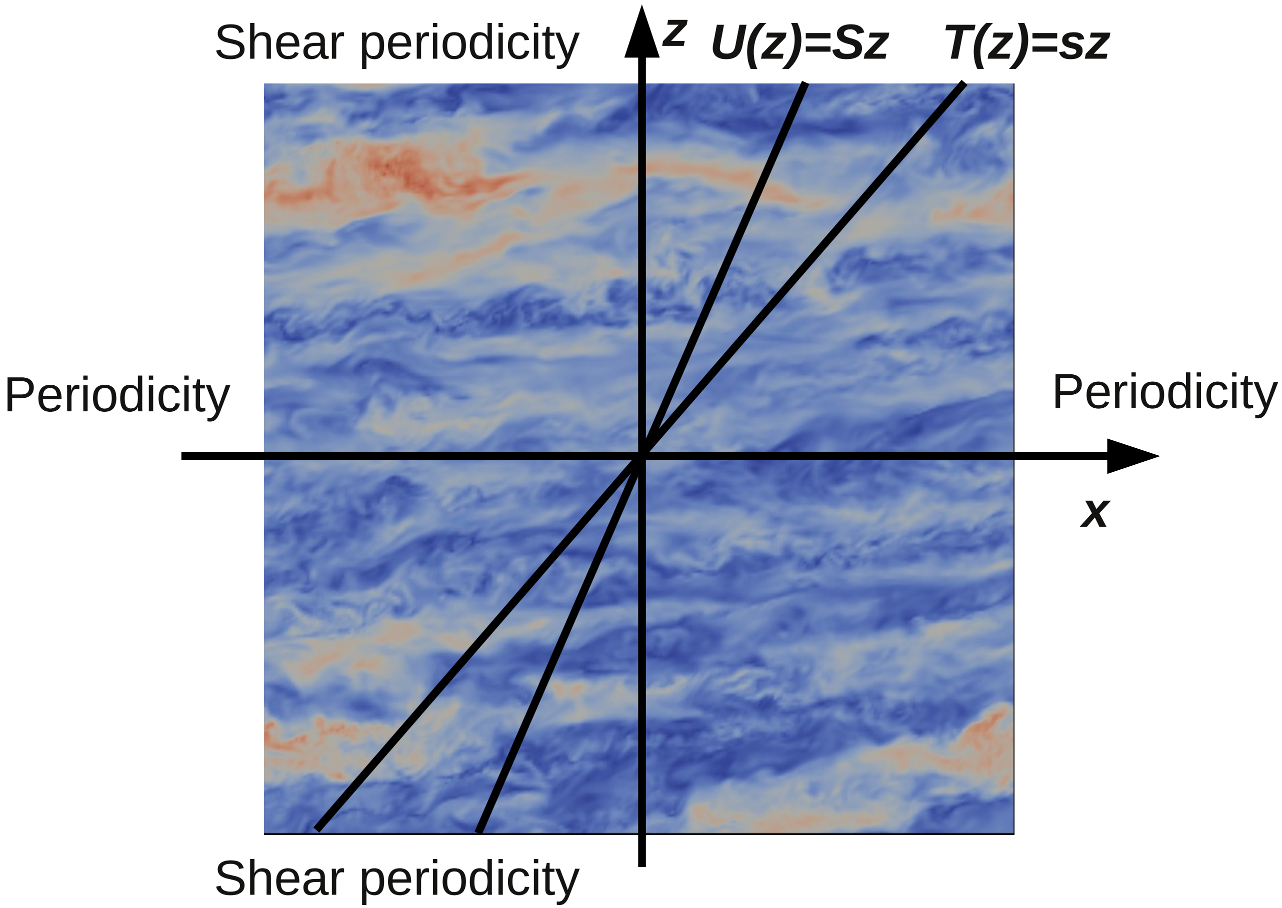
- **Numerical weather prediction models need parameterizations (models) for it**
- **Current parameterizations are still based on partially inadequate understanding and calibration data of stably stratified turbulence**
- **Weather and air quality forecasts as well as climate studies could benefit from better parameterizations for stably stratified turbulence**
- **Understanding how nature works is valuable as such**

Background and motivation

Some challenges for numerical methods:

- **Only a narrow band of atmospheric scales can be resolved in simulations → at least one cut-off in the spectrum of motion must be handled somehow**
- **High numerical accuracy is needed as the smallest resolved length scales are typically represented by a few nodes only**
- **Discretization schemes employed for each term should preferably have the same conservation properties as these terms in the non-discretized Navier-Stokes equations**
- **In the problem of homogeneous shear turbulence the mean shear is decoupled from the turbulent motion (idealization) and this leads to some specific requirements for the numerical methods**

Computational method: setup



Computational method: shear periodicity

As the mean velocity *U(z)*=*Sz*, the shear periodicity of any variable *Φ* is defined as

$$\phi(x, y, z+L_z)=\phi(x-Stz, y, z)$$

Interpolations are done in Fourier space (Gerz et al., 1989).

Computational method: equations

- **Incompressible homogeneous turbulence under constant mean shear and stratification**
- **Mean velocity and temperature uncoupled with fluctuations**
- **Boussinesq approximation for density fluctuations**
- **Equations non-dimensionalized by *ΔU, ΔT, L/2π:***

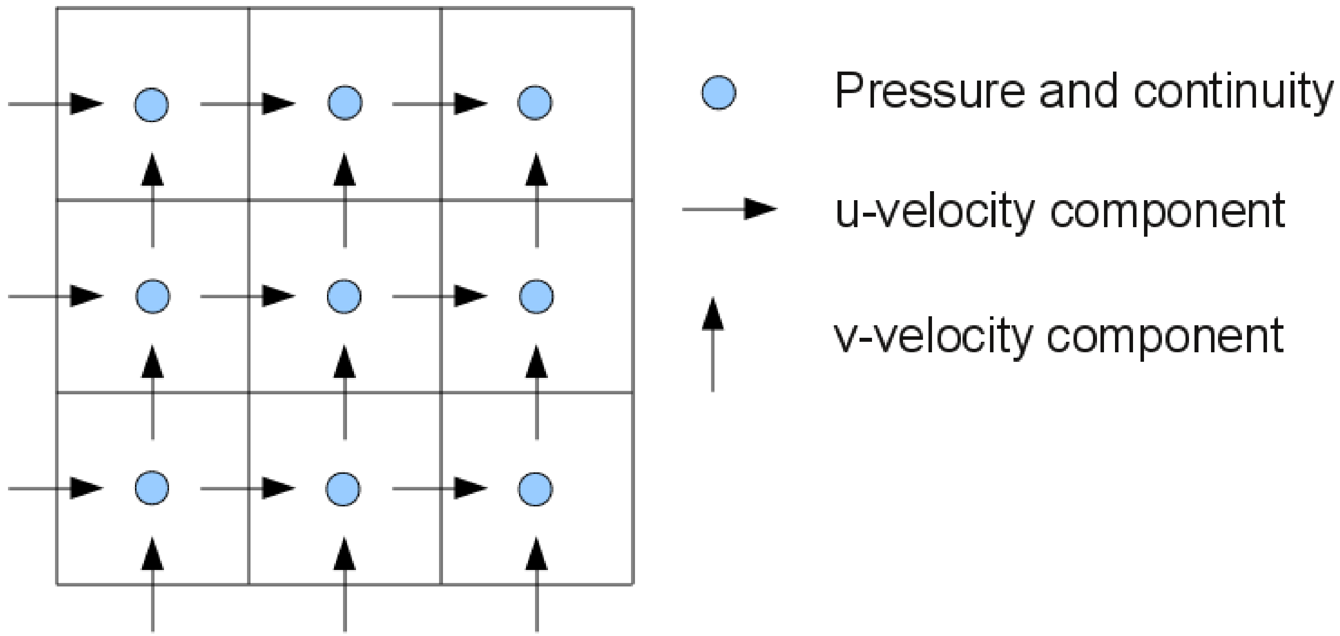
$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} + Sx_3 \frac{\partial u_i}{\partial x_1} + \frac{\partial p}{\partial x_i} = \frac{2\pi}{Re_L} \frac{\partial^2 u_i}{\partial x_j \partial x_j} - u_3 S \delta_{i1} + RiS\theta\delta_{i3}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \theta u_j}{\partial x_j} + Sx_3 \frac{\partial \theta}{\partial x_1} = \frac{2\pi}{Re_L Pr} \frac{\partial^2 \theta}{\partial x_j \partial x_j} - u_3 s$$

Computational method: numerics overview

- **Spatial discretization by 6th-order fully conservative finite-difference scheme (Morinishi, 1998)**
- **Time integration explicit 3rd-order Runge-Kutta**
- **Fractional step method with mean-advection step and pressure-projection step**
- **Mean-advection step by Fourier interpolation**
- **Poisson equation for pressure is solved in Fourier space to avoid iteration (with modified wave numbers)**
- **An extra transformation is done inside the 3-D FFT to handle the shear periodicity (Gerz et al. used a more complicated approach which combined FFTs and Gaussian elimination)**

Computational method: grid arrangement



Staggered grid arrangement (only two dimensions shown)

Computational method: some definitions

Define the following finite-difference and interpolation operators (shown only in x_1-direction for example)

$$\left(\frac{\delta_n \phi}{\delta_n x_1} \right)_{x_1, x_2, x_3} \equiv \frac{\phi(x_1 + nh_1/2, x_2, x_3) - \phi(x_1 - nh_1/2, x_2, x_3)}{nh_1}$$

$$\left(\overline{\phi}^{nx_1} \right)_{x_1, x_2, x_3} \equiv \frac{\phi(x_1 + nh_1/2, x_2, x_3) + \phi(x_1 - nh_1/2, x_2, x_3)}{2}$$

h_1 is the (constant) grid spacing in x_1-direction

ILMATIETEEN LAITOS
METEOROLOGISKA INSTITUTET
FINNISH METEOROLOGICAL INSTITUTE

Computational method: advection scheme

The advection terms in the divergence form are approximated at the momentum nodes using Morinishi's 6th-order fully conservative advection scheme

$$\frac{\partial u_j u_i}{\partial x_j} \approx \frac{150}{128}\frac{\delta_1}{\delta_1 x_j}\left[\left(\frac{150}{128}\overline{u}_j^{1x_i} - \frac{25}{128}\overline{u}_j^{3x_i} + \frac{3}{128}\overline{u}_j^{5x_i}\right)\overline{u}_i^{1x_j}\right]$$
$$- \frac{25}{128}\frac{\delta_3}{\delta_3 x_j}\left[\left(\frac{150}{128}\overline{u}_j^{1x_i} - \frac{25}{128}\overline{u}_j^{3x_i} + \frac{3}{128}\overline{u}_j^{5x_i}\right)\overline{u}_i^{3x_j}\right]$$
$$+ \frac{3}{128}\frac{\delta_5}{\delta_5 x_j}\left[\left(\frac{150}{128}\overline{u}_j^{1x_i} - \frac{25}{128}\overline{u}_j^{3x_i} + \frac{3}{128}\overline{u}_j^{5x_i}\right)\overline{u}_i^{5x_j}\right]$$

Computational method: diffusion scheme

- **Second derivatives for the viscous terms are approximated at the momentum nodes using a standard 6th-order scheme**

$$\frac{\partial^2 u_i}{\partial^2 x_j} \approx -\frac{490}{180\, h_1^2} u_i(x_1) + \frac{270}{180\, h_1^2} [u_i(x_1 - h_1) + u_i(x_1 + h_1)]$$

$$-\frac{27}{180\, h_1^2} [u_i(x_1 - 2h_1) + u_i(x_1 + 2h_1)]$$

$$+\frac{2}{180\, h_1^2} [u_i(x_1 - 3h_1) + u_i(x_1 + 3h_1)]$$

here x_2 and x_3 are dropped in order to save space

Computational method: continuity & pressure

In addition we need to discretize the continuity equation at the pressure nodes

$$\frac{\partial u_j}{\partial x_j} \approx D(\vec{u}) \equiv \frac{150}{128}\frac{\delta_1 u_j}{\delta_1 x_j} - \frac{25}{128}\frac{\delta_3 u_j}{\delta_3 x_j} + \frac{3}{128}\frac{\delta_5 u_j}{\delta_5 x_j} = 0$$

as well as the pressure gradient

$$\frac{\partial p}{\partial x_i} \approx G_i(p) \equiv \frac{150}{128}\frac{\delta_1 p}{\delta_1 x_i} - \frac{25}{128}\frac{\delta_3 p}{\delta_3 x_i} + \frac{3}{128}\frac{\delta_5 p}{\delta_5 x_i}$$

The discrete operators *D* and *G* will be used later

Computational method: fractional stepping

- **each Runge-Kutta step consists of a predictor step and two corrective steps: the mean advection step and the projection step (Chorin, 1968)**
- **predictor step integrates the equations without the mean-advection and pressure-gradient terms to obtain the first intermediate fields u'_i and θ'**
- **the first intermediate fields are then simply moved (slid) according to the linear mean-velocity advection using Fourier interpolation similarly to Gerz et al. (1989) to obtain the second intermediate velocity field u^*_i and the final temperature field θ**

Computational method: fractional stepping

- **u^*_i-field is then projected onto divergence-free field**
- **the projection is based on the following discrete Poisson-equation for a pressure-like projection variable *p'***

$$D(G(p'))=\frac{1}{\alpha\,\Delta t}D(\vec{u}*)$$

- ***p'* is solved in the wave-number space using wave numbers modified according to the discrete Laplace operator *D(G)* and also according to transformation to a shear-periodic coordinates**
- **finally the projection is made as**

$$u_i=u_i*-\alpha\,\Delta t\,G_i(p')$$

Computational method: time integration

Explicit third-order 2-N low-storage Runge-Kutta method (Williamson, 1980) is employed for the time integration

- **2-N low-storage methods need only two full-size arrays for each transport equation: one for the variable itself and another for the right-hand side**
- **semi-implicit fractional-step methods which integrate the viscous terms by the Crank-Nicholson scheme are only first order accurate in time**
- **but the present fully explicit method avoids this shortcoming and retains the 3rd-order accuracy of the time-integration scheme**
- **time steps need to be small but this is good for accuracy**

Computational method: implementation

- **Development started on March 2009**
- **Coding in Fortran 90**
- **Only FFTs by 'library' routines (open-source fftpack developed at NCAR)**
- **The code is completely verified by the method of manufactured solutions (MMS)**
- **Simple open-MP parallelization allows using 12 processing elements (PE) for a simulation on our Cray XT5 as each of its nodes have 12 PEs sharing a common memory**
- **Speed-up from this 12-PE Open-MP parallelization is unfortunately rather modest**
- **MPI-parallelization is not yet done :-(**

Theoretical aspects

Billant & Chomaz (2001): $L_v \sim q/N << L_h$, $L_v/L_h \sim Fr_h << 1$ and $w << u \sim v$ (only theoretical analysis).

Brethouwer et al. (2007): strongly stratified turbulence has two regimes: $R = Fr_h^2 Re > 1$ and $R < 1$.

Quasi horizontal layers i.e. "pancakes" are formed.

Brethouwer et al. (2007) made DNS but without mean shear. They used artificial forcing of the lowest horizontal Fourier modes to sustain stationary energy.

Lindborg (2006) and Waite & Bartello (2006) used forcing as well in their large-scale hyperviscosity simulations.

Is the dynamics similar also with vertical mean shear (are the pancakes still there and if so, are they similar)?

Theoretical aspects

Energy equations: kinetic, potential and total energies. Potential energy is defined as $E_p=Ri\langle\theta^2\rangle/2$.

$$\frac{\mathrm{d}E_k}{\mathrm{d}t}=-\langle uw\rangle S-\varepsilon_k+RiS\langle w\theta\rangle$$

$$\frac{\mathrm{d}E_p}{\mathrm{d}t}=-\varepsilon_p-RiS\langle w\theta\rangle$$

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\langle uw\rangle S-\varepsilon_k-\varepsilon_p$$

with

$$\varepsilon_k=\frac{2\pi}{Re}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} \quad \text{and} \quad \varepsilon_p=\frac{2\pi Ri}{Re\,Pr}\frac{\partial\theta}{\partial x_j}\frac{\partial\theta}{\partial x_j}$$

Theoretical aspects

With artificial forcing, sufficient energy is supplied to maintain constant energy. In this situation energy always flows from kinetic to potential energy.

This implies that $\langle w\theta \rangle$ remains negative (no counter-gradient buoyancy fluxes).

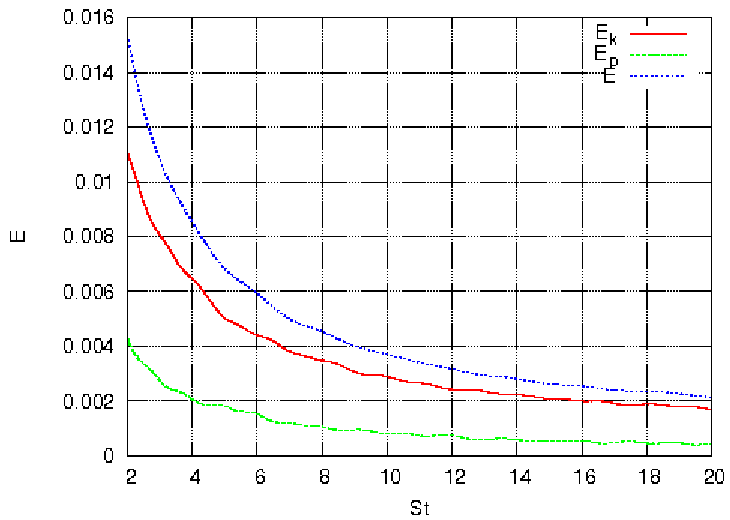
With mean shear and no forcing buoyancy flux $RiS\langle w\theta \rangle$ oscillates between negative and positive values i.e. potential energy is intermittently transferred back to kinetic energy via $\langle ww \rangle$.

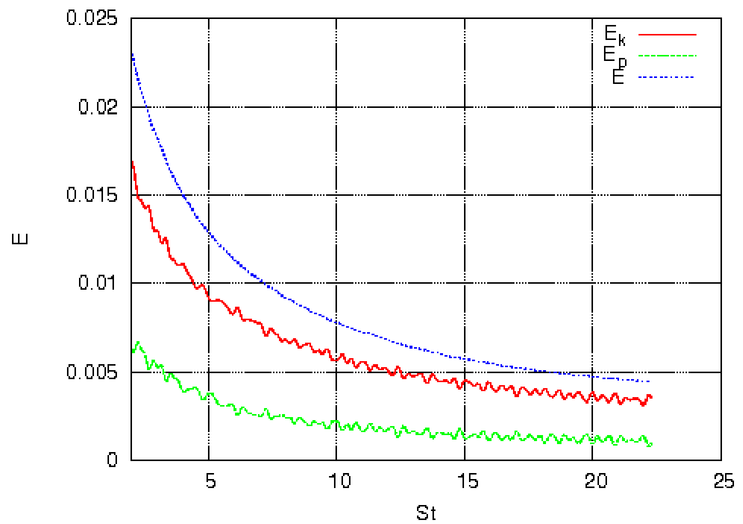
Therefore $\langle ww \rangle$ is larger than in case of forcing.

Moreover $|\langle uw \rangle|$ goes small and $\langle uw \rangle$ is intermittently positive (negative production → energy decays).

The vertical length scale of w no more scales with q/N while that of u and v still seem to scale with q/N

Results: energetics

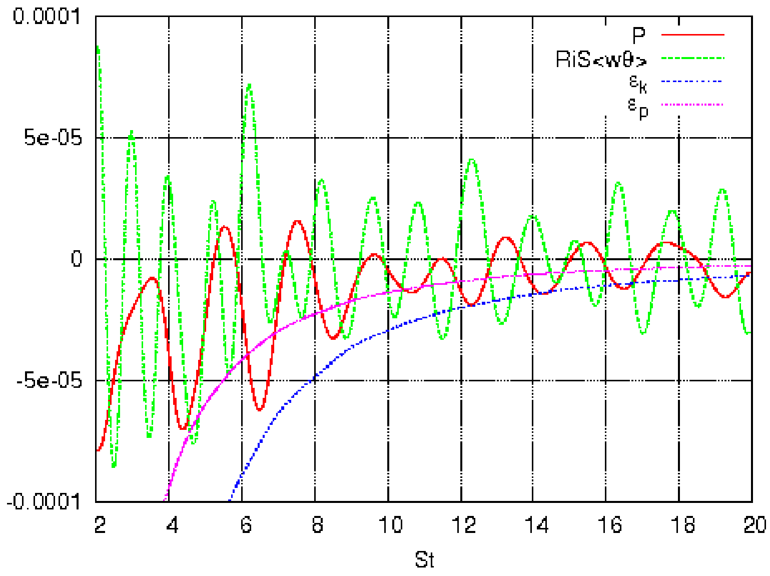


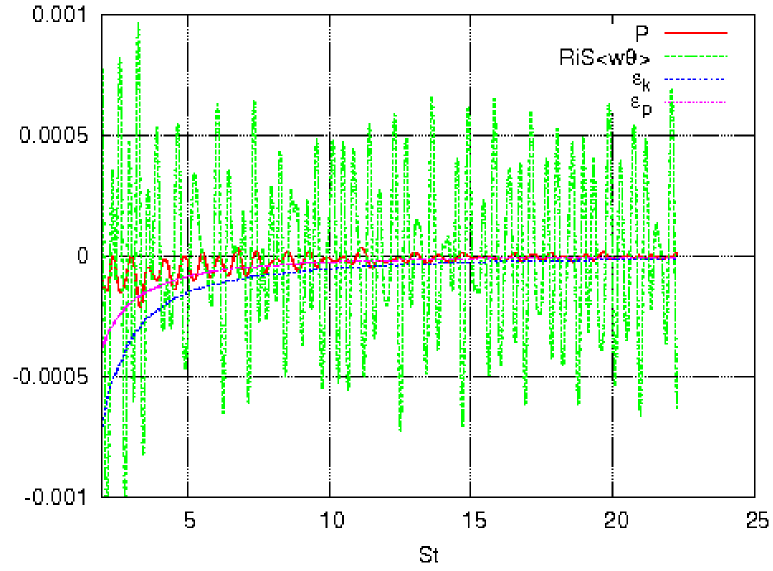


Evolution of kinetic, potential and total energies.

Left: *Ri*=10, right: *Ri*=100.

Results: energetics

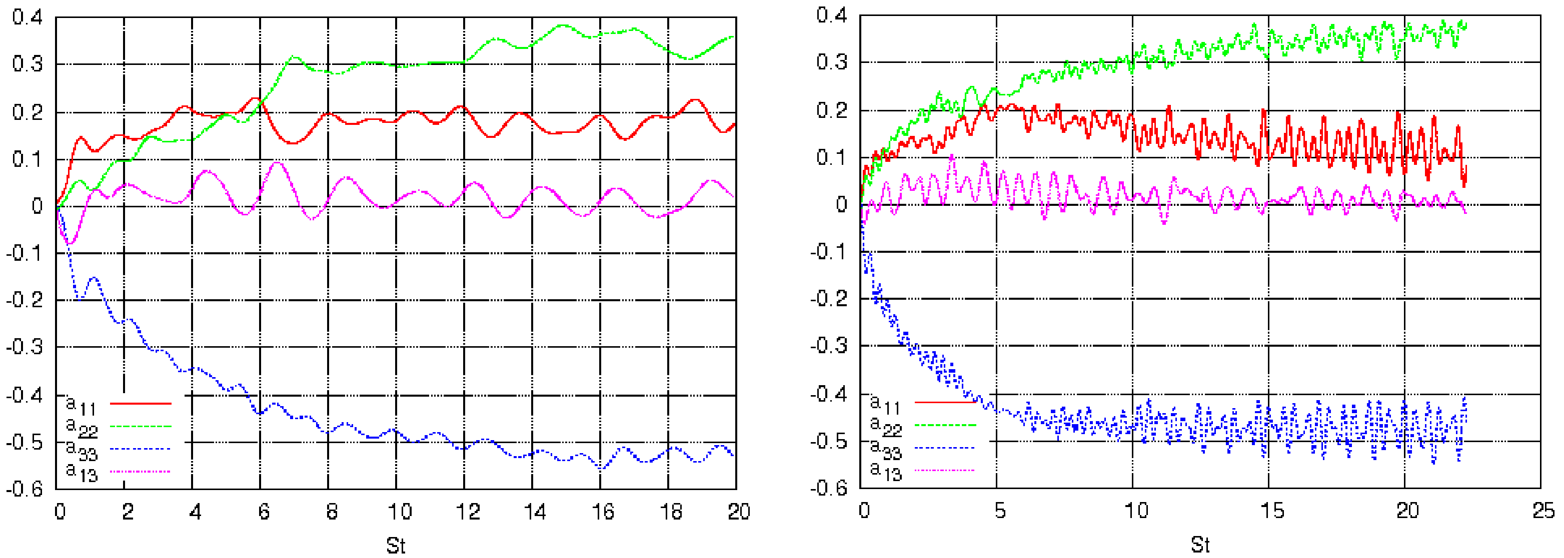




Energy budgets: shear production, buoyancy flux, dissipations of kinetic and potential energy.

Left: *Ri*=10, right: *Ri*=100.

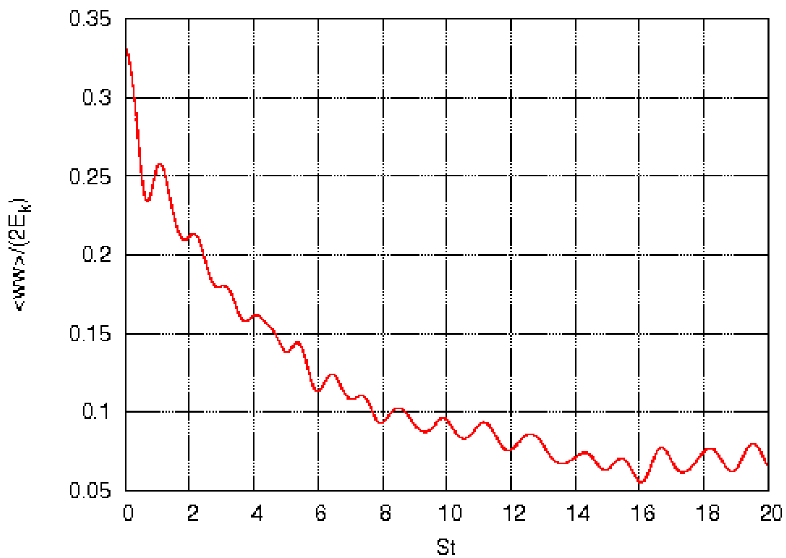
Results: anisotropy

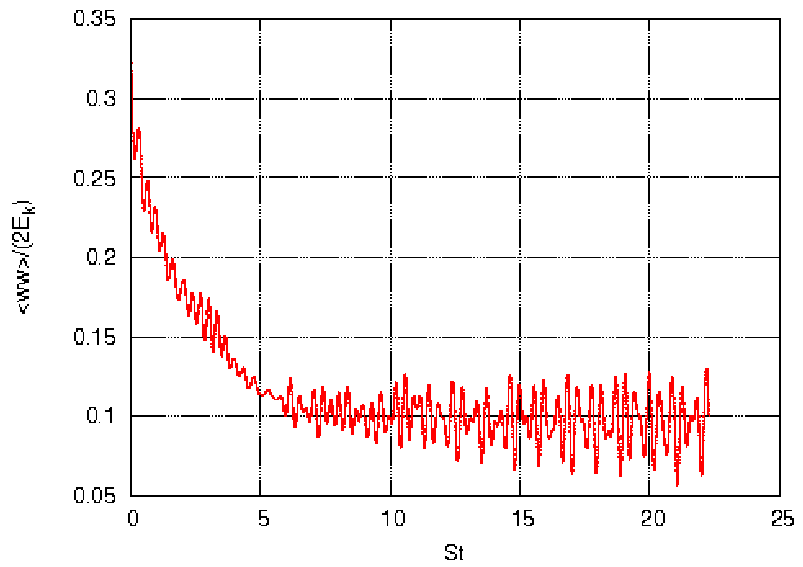


Reynolds stress anisotropies $a_{ij} = (\langle u_i u_j \rangle - 2/3 E_k \delta_{ij})/E_k$

Left: *Ri*=10, right: *Ri*=100.

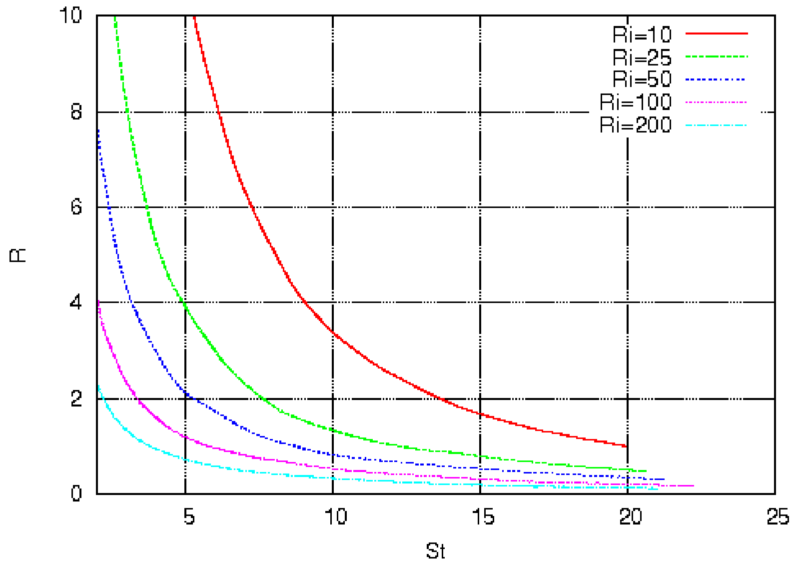
Results: anisotropy

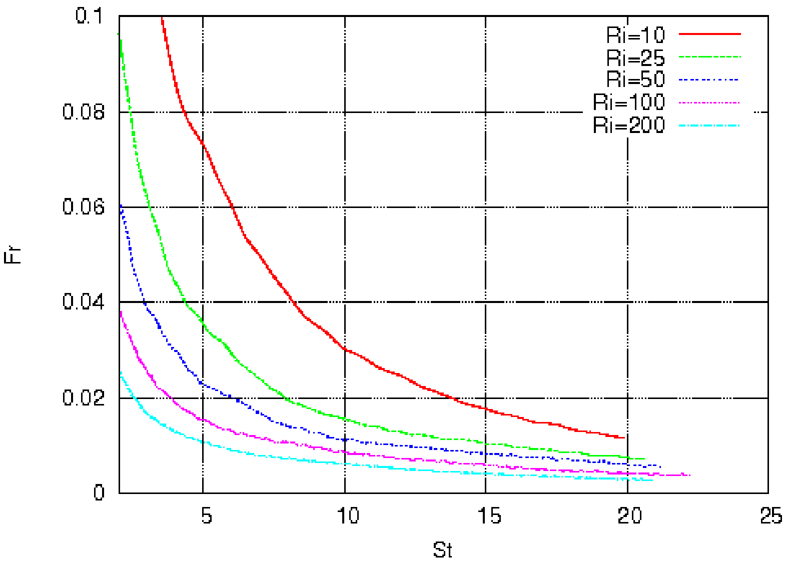




Vertical velocity variance <*ww*> scaled by $2E_k$.
Brethouwer et al. have roughly one order of magnitude lower values!

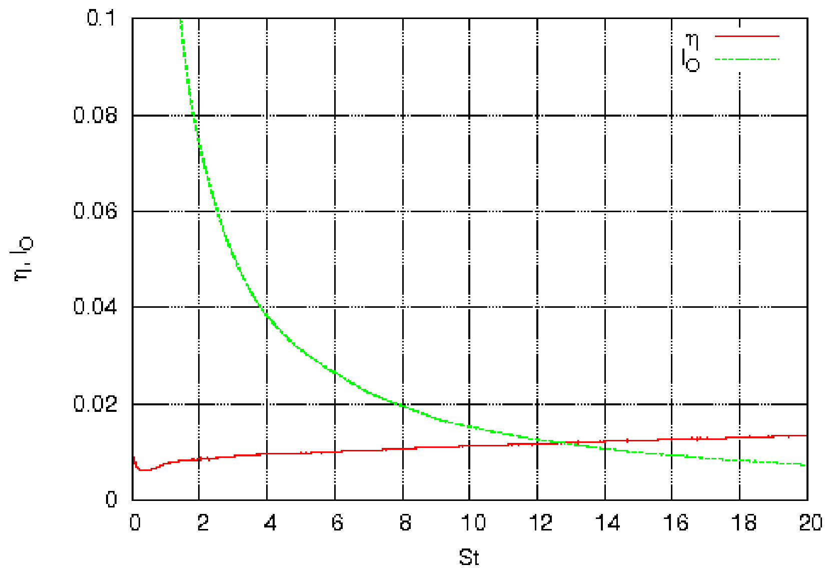
Results: evolution of R and Fr_h

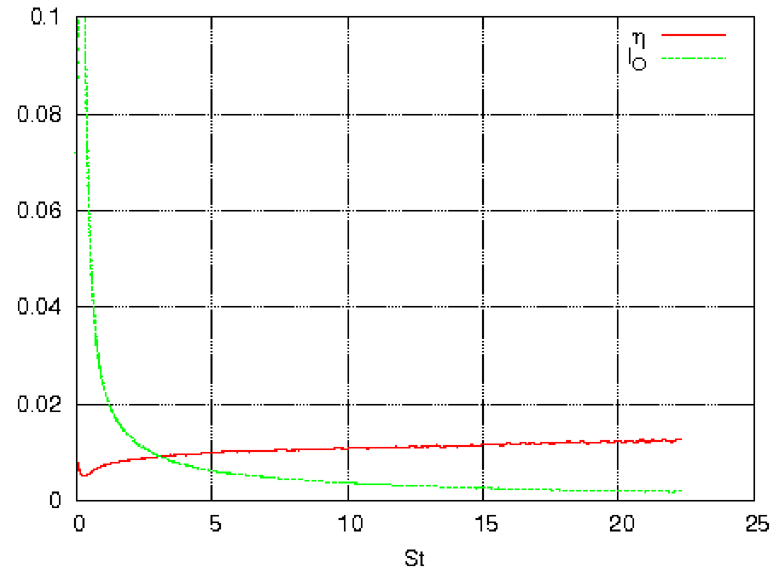




Both *R* and Fr_h decrease in time but *Re* typically slowly increases.

Results: Kolmogorov and Ozmidov scales



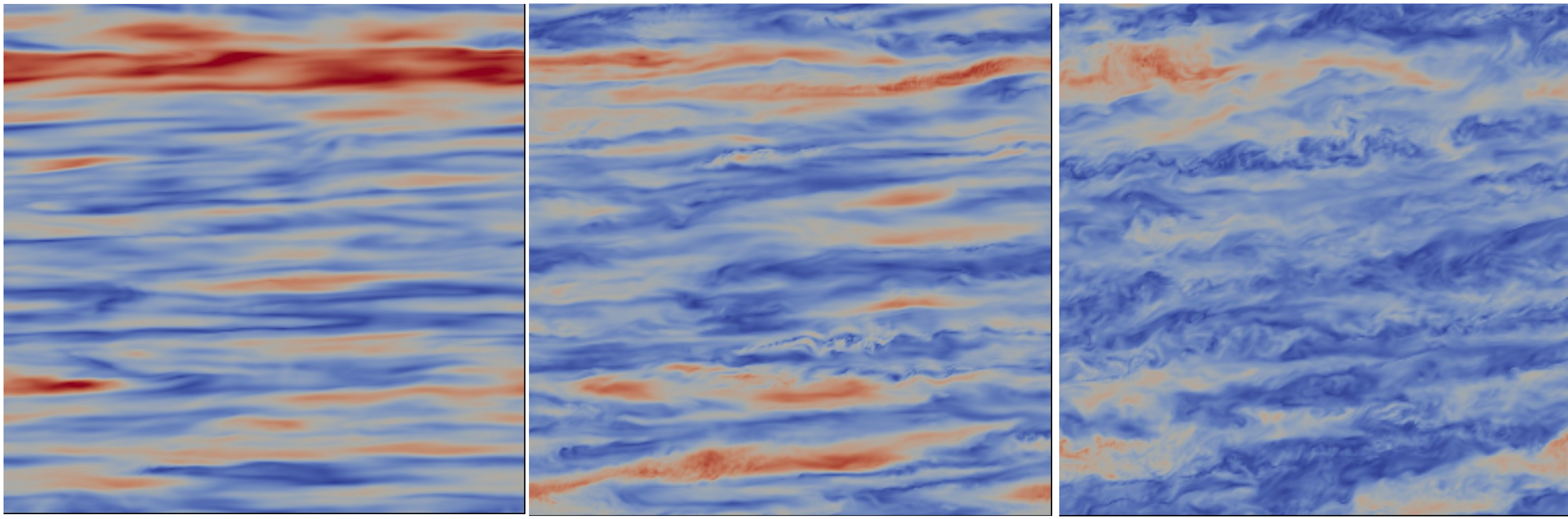


Kolmogorov length $\eta = (\nu^3/\varepsilon)^{1/4}$ and

Ozmidov length $l_O = (\varepsilon/N^3)^{1/2}$ as functions of time.

Left: Ri = 10 and right: Ri = 100.

Results: the effect of R and Fr_h



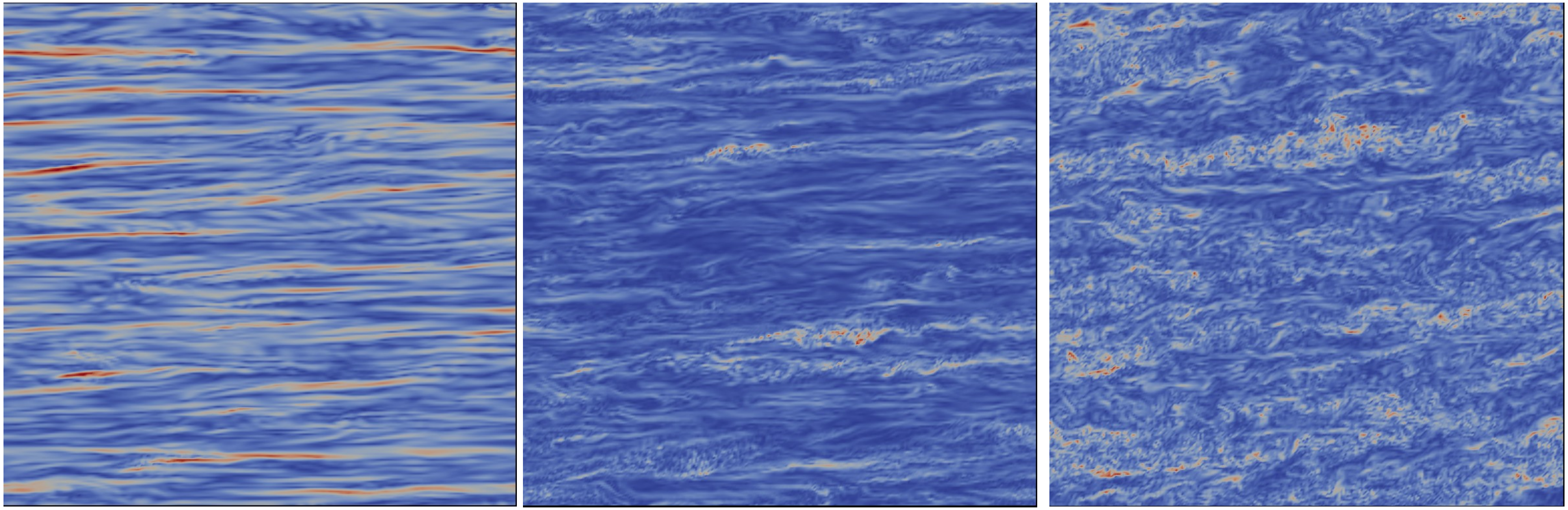
|V|. R, Fr_h and q/N increasing from left to right:

a) $R = 0.31$, $Fr_h = 0.0058$, $Re = 8900$, $q/N = 0.053$

b) $R = 1.15$, $Fr_h = 0.014$, $Re = 5490$, $q/N = 0.095$

c) $R = 7.22$, $Fr_h = 0.053$, $Re = 2500$, $q/N = 0.182$

Results: the effect of R and Fr_h



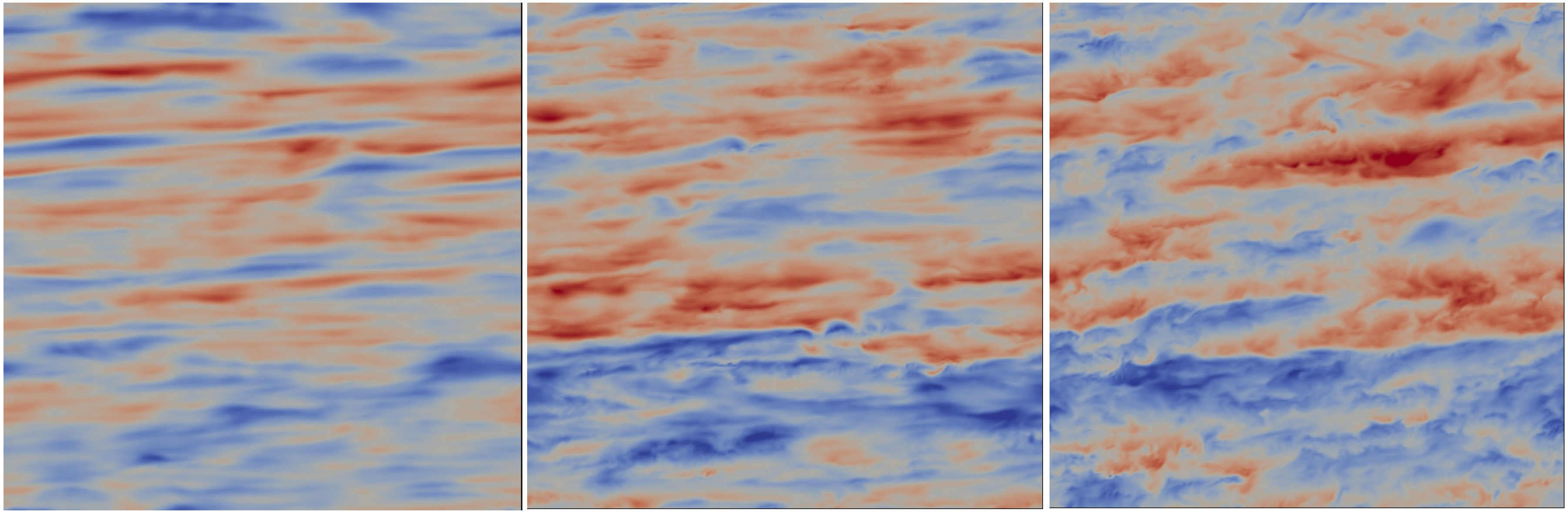
Vorticity. R, Fr_h and q/N increasing from left to right:

a) $R = 0.31$, $Fr_h = 0.0058$, $Re = 8900$, $q/N = 0.053$

b) $R = 1.15$, $Fr_h = 0.014$, $Re = 5490$, $q/N = 0.095$

c) $R = 7.22$, $Fr_h = 0.053$, $Re = 2500$, $q/N = 0.182$

Results: the effect of R and Fr_h



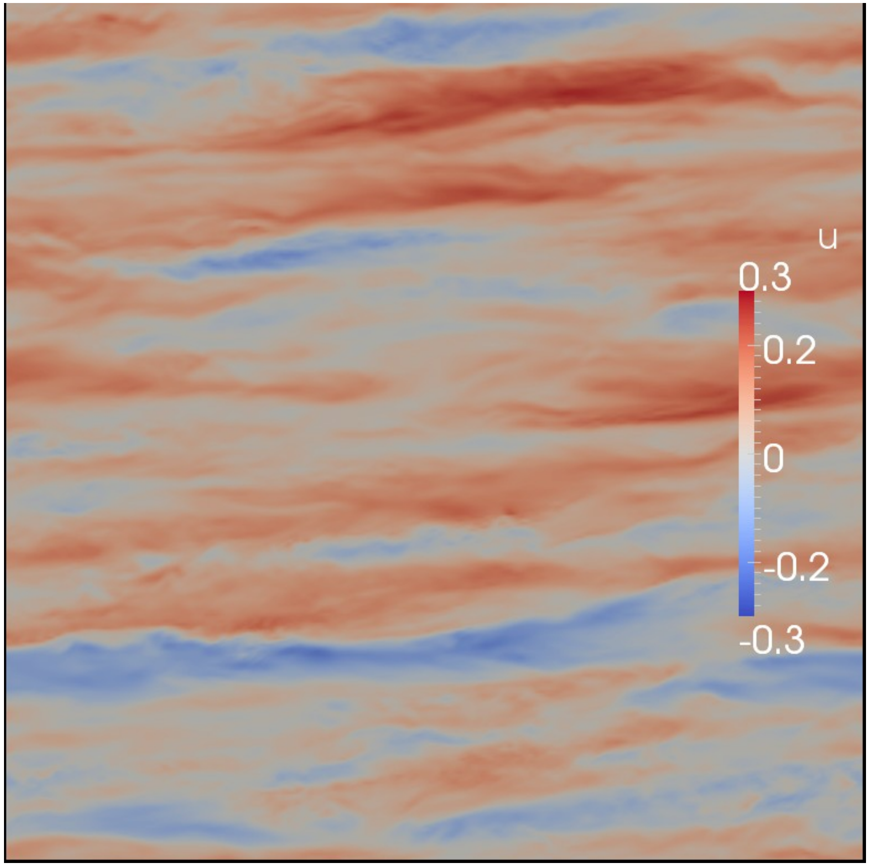
Temperature. R, Fr_h and q/N increasing from left to right:

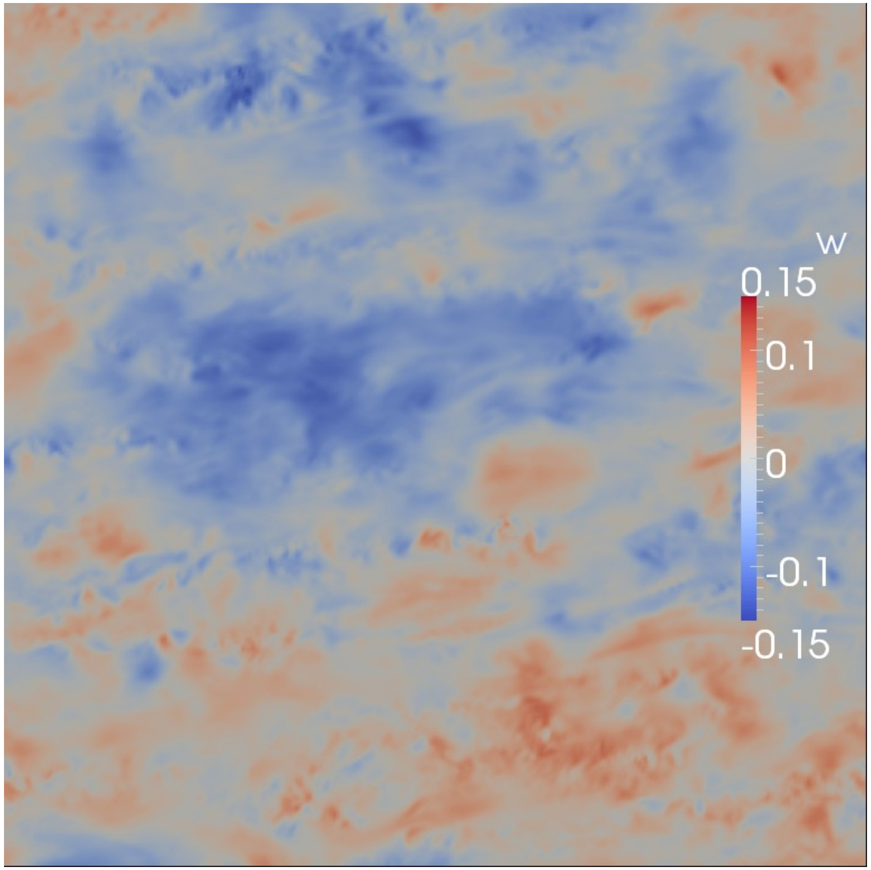
a) $R = 0.31$, $Fr_h = 0.0058$, $Re = 8900$, $q/N = 0.053$

b) $R = 1.15$, $Fr_h = 0.014$, $Re = 5490$, $q/N = 0.095$

c) $R = 7.22$, $Fr_h = 0.053$, $Re = 2500$, $q/N = 0.182$

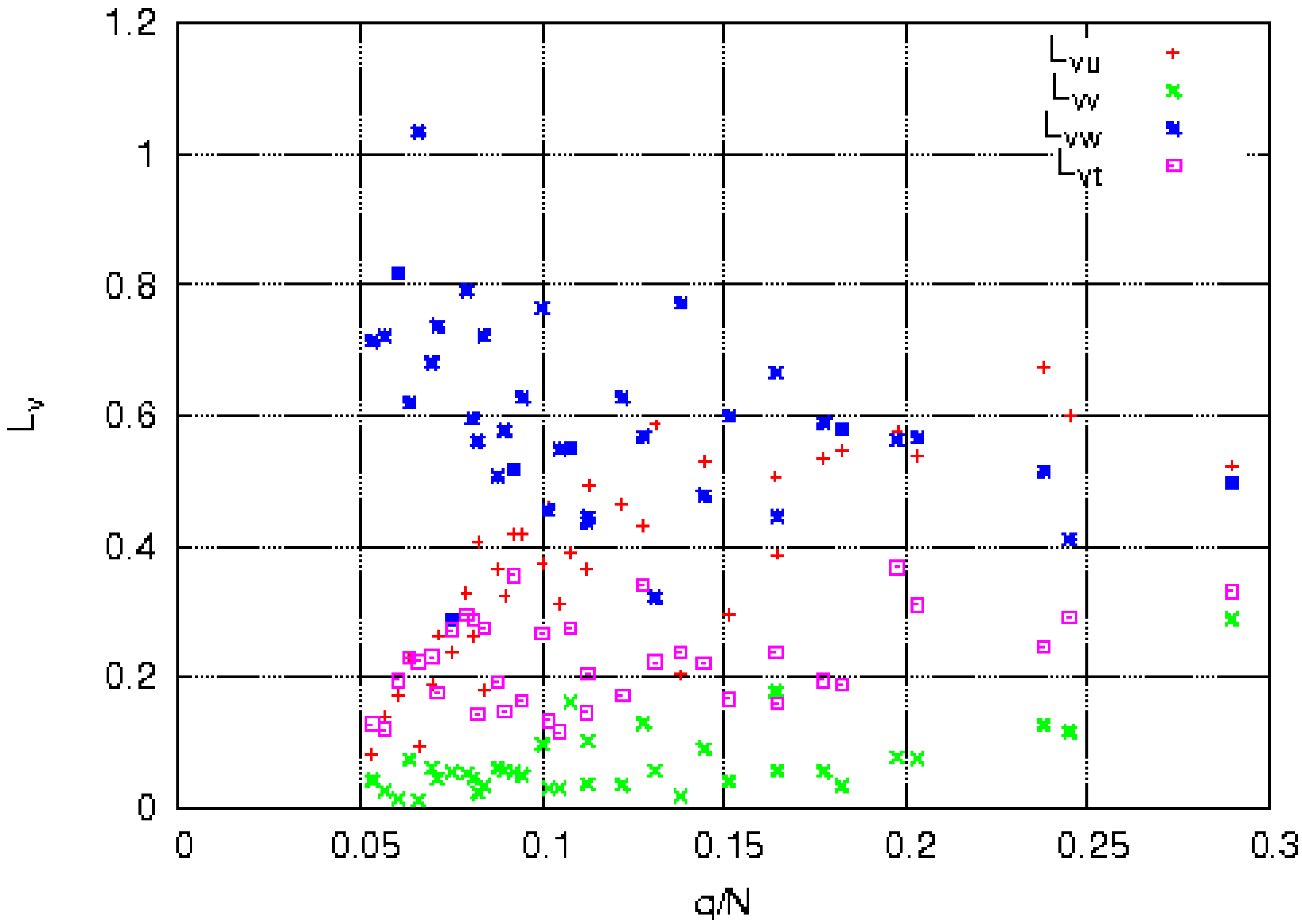
Results: velocity fields *u* and *w*



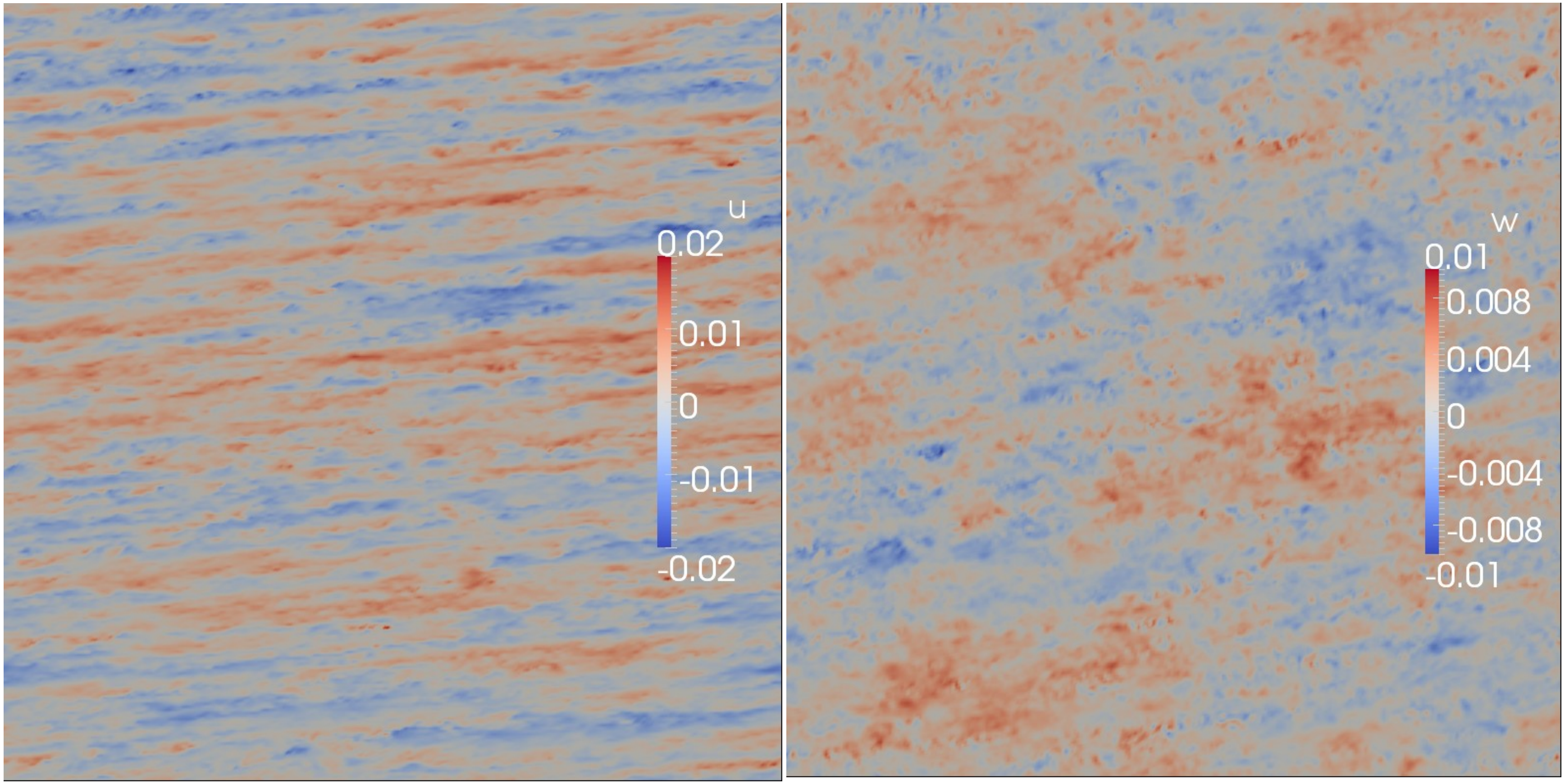


***R*=1.67, Fr_h=0.019, large initial $L_h=E_k^{3/2}/\varepsilon$, *Ri*=50, *St*=6.2**

Results: vertical length scales, large initial L_h

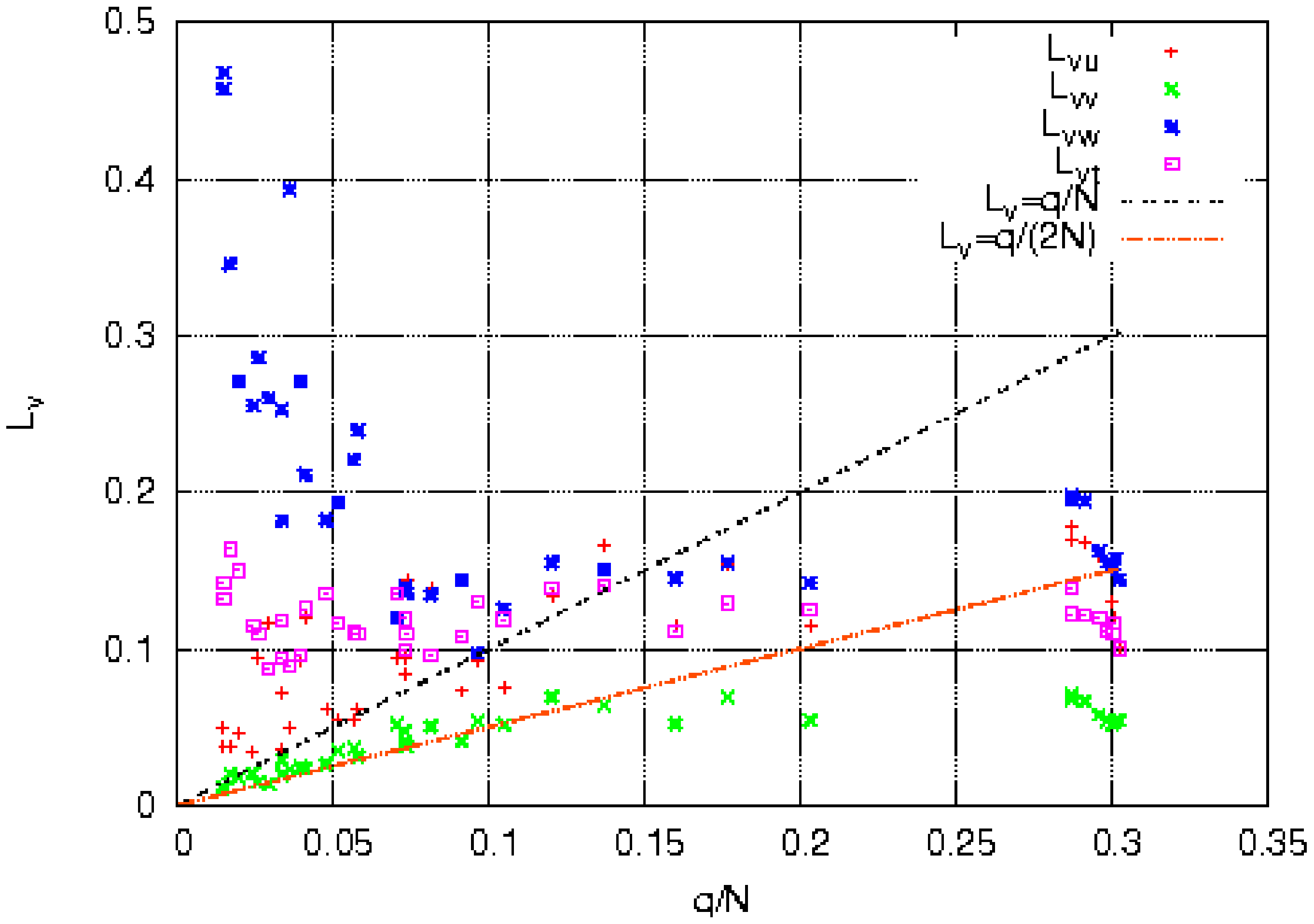


Results: velocity fields *u* and *w*



R=2.72, Fr_h=0.17, small initial $L_h=E_k^{3/2}/\varepsilon$, Ri=1, St=12

Results: vertical length scales, small initial L_h



Preliminary conclusions and outlook

- **Counter-gradient fluxes, intermittent energy transfer (kinetic ↔ potential) and negative shear production are characteristic to strongly stratified *sheared non-forced* turbulence**
- **Vertical velocity fluctuations are not as small as in *forced non-sheared* stratified turbulence**
- **Vertical length scale of vertical velocity structures seems to not be small and does not scale by *q/N***
- **Thus, stratified *sheared non-forced* turbulence seems slightly different from stratified *forced non-sheared* turbulence**
- **However, there seems to be more common features than differences between these two cases**
- ***Forced and sheared* cases are to be studied next**

ILMATIETEEN LAITOS
METEOROLOGISKA INSTITUTET
FINNISH METEOROLOGICAL INSTITUTE

Acknowledgements

This work is part of the European Research Council funded project: Atmospheric planetary boundary layers: physics, modelling and role in earth system

PBL-PMES

(grant agreement no 227915) led by Research Professor Sergej Zilitinkevich

It has also received support from the Academy of Finland (grant agreement no 217537) and from EU FP7 project MEGAPOLI (grant agreement no 212520)

Some definitions

$$Ri = \frac{N^2}{S^2} = \frac{g}{T_0}\frac{s}{S^2} \quad \text{with} \quad S = \frac{\mathrm{d}U}{\mathrm{d}z} \quad \text{and} \quad s = \frac{\mathrm{d}T}{\mathrm{d}z}$$

$$N = \left(\frac{gs}{T_0}\right)^{1/2}$$

$$q = \left(\langle uu\rangle + \langle vv\rangle + \langle ww\rangle\right)^{1/2} = \left(2E_k\right)^{1/2}$$

$$Re = \frac{E_k^2}{\varepsilon\nu} = \frac{L_T E_k^{1/2}}{\nu} \quad \text{and} \quad Re_L = \frac{\Delta U L}{\nu}$$

$$Fr_h = \frac{q}{NL_h} = \frac{\varepsilon}{Nq^2}$$

$$R = Fr^2 Re = \frac{\varepsilon}{\nu N^2}$$

Some references

Billant, P. & Chomaz, J.-M., 2000a: Experimental evidence for a new instability of a vertical columnar vortex pair in a strongly stratified fluid. J. Fluid Mech. 418, 167-188.

Billant, P. & Chomaz, J.-M., 2000b: Theoretical analysis of the zigzag instability of a vertical columnar vortex pair in a strongly stratified fluid. J. Fluid Mech. 419, 29-63.

Billant, P. & Chomaz, J.-M., 2001: Self-similarity of strongly stratified inviscid flows. Phys. Fluids 13, 1645-1651.

Brethouwer, G. & Billant, P. & Lindborg, E. & Chomaz, J.-M., 2007, Scaling analysis and simulation of strongly stratified turbulent flows. J. Fluid Mech. 585, 343-368.

Chorin, A.J., 1968. Numerical solution of the Navier-Stokes equations. Mathematics of Computation 22, 745-762.

Gerz, T., Schumann, U. & Elghobashi, S.E., 1989: Direct numerical simulation of stratified homogeneous turbulent shear flows. J. Fluid Mech. 200, 563-594.

Herring, J.R. & Metais, O., 1989: Numerical experiments in forced stably stratified turbulence. J. Fluid Mech. 202, 97-115.

Holt, S.E., Koseff, J.R., Ferziger, J.H., 1992: A numerical study of the evolution and structure of homogeneous stably stratified sheared turbulence. J. Fluid Mech. 237, 499-539.

Some references

Lindborg, E., 2006: The energy cascade in a strongly stratified fluid. J. Fluid. Mech. 550, 207-242.

Morinishi, Y. & Lund, T.S. & Vasilyev, O.V., Moin, P., 1998: Fully conservative higher order finite difference schemes for incompressible flow. J. Comp. Phys. 143, 90-124.

Riley, J.J., Metcalfe, R. & Weissman, M.A., 1981: Direct numerical simulations of homogeneous turbulence in density stratified fluids. In *proc. AIP Conf. on Nonlinear Properties of Internal Waves*, 79-112. Woodbury.

Waite, M.L. & Bartello, P., 2006: The transition from geostrophic to stratified turbulence. J. Fluid Mech. 568, 89-108.

Williamson, J.H., 1980. Low-storage Runge-Kutta Schemes. Journal of Computational Physics 35, 48.