Elmer capabilities in acoustics

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Elmer User Meeting, May 29, 2006

0. Introduction

The outline of the presentation:

1. Mathematical models
   1.1 Linearized Navier-Stokes equations
   1.2 Further approximations
2. Examples
3. Concluding remarks

1. Mathematical models

1.1 Linearized Navier-Stokes equations

The most extensive acoustical model available in Elmer characterizes the flow of a viscous and thermally conducting fluid.

To introduce this model, we introduce the following notations:

- the velocity, density, pressure and temperature fields associated with the fluid flow are denoted by $\mathbf{V}(\mathbf{x},t)$, $\rho(\mathbf{x},t)$, $p(\mathbf{x},t)$ and $T(\mathbf{x},t)$, respectively, with $\mathbf{x}$ and t being place and time

- the values of the density, pressure and temperature at an equilibrium state are denoted by ρ_0, p_0 and T_0

- the outward unit normal vector to the boundary of a body Ω which is occupied by the fluid is denoted by $\mathbf{e}_n$, and $\mathbf{e}_s$ and $\mathbf{e}_t$ are mutually orthogonal unit vectors tangential to the boundary.

The full model is based on the field equations

$$
\begin{aligned}
\rho_0 \frac{\partial \mathbf{V}}{\partial t} &= \operatorname{div} \mathbf{T} + \rho_0 \mathbf{b}, \\
\mathbf{T} &= -p\mathbf{I} + \lambda(\operatorname{div} \mathbf{V})\mathbf{I} + 2\mu \mathbf{D}(\mathbf{V}), \\
\mathbf{D}(\mathbf{V}) &= \frac{1}{2}(\operatorname{grad} \mathbf{V} + \operatorname{grad} \mathbf{V}^T), \\
\frac{\partial \rho}{\partial t} &= -\rho_0 \operatorname{div} \mathbf{V}, \\
\rho_0 \frac{du}{dt} &= \kappa \operatorname{div} \operatorname{grad} T - p_0 \operatorname{div} \mathbf{V} + \rho_0 h,
\end{aligned}
\tag{1}
$$

where $\mathbf{T}$ and $\mathbf{I}$ are the stress and identity tensor, $\mathbf{b}$ is the body force (per unit mass), λ and μ are parameters characterizing the viscosity of the fluid, u is the specific internal energy, κ is the heat conductivity and h is the internal supply of heat.

We denote the specific entropy (entropy per unit mass) and its equilibrium value by s and s_0 and assume that the relation

$$du = T_0 ds + (p_0/\rho_0^2) d\rho \tag{2}$$

is valid. In addition, we approximate the equations which give the changes of pressure and specific entropy in terms of the changes of the state variables by

$$p - p_0 = \frac{(\gamma - 1)\rho_0 C_V}{T_0 \beta}(T - T_0) + \frac{(\gamma - 1) C_V}{T_0 \beta^2}(\rho - \rho_0) \tag{3}$$

and

$$s - s_0 = \frac{C_V}{T_0}(T - T_0) - \frac{C_V(\gamma - 1)}{T_0 \rho_0 \beta}(\rho - \rho_0), \tag{4}$$

where C_V is the specific heat at constant volume (per unit mass), γ is the ratio of the specific heats at constant pressure and constant volume and β is the coefficient of thermal expansion defined by

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p. \tag{5}$$

- The solutions of the primary unknowns are assumed to be of the form

$$\begin{aligned} \mathbf{V}(\mathbf{x},t) &= \mathbf{V}(\mathbf{x})\exp(i\omega t), \\ \rho(\mathbf{x},t) &= \rho_0 + \rho(\mathbf{x})\exp(i\omega t), \\ T(\mathbf{x},t) &= T_0 + T(\mathbf{x})\exp(i\omega t), \end{aligned} \tag{6}$$

where i is the imaginary unit and ω is the angular frequency.

- The field equations reduce then to

$$\begin{aligned} i\omega\rho_0\mathbf{V} + \frac{(\gamma-1)C_V\rho_0}{\beta T_0}\operatorname{grad} T - (\lambda + \mu - \frac{i(\gamma-1)C_V\rho_0}{\omega T_0\beta^2})\operatorname{grad}\operatorname{div}\mathbf{V} - \mu\operatorname{div}\operatorname{grad}\mathbf{V} &= \rho_0\mathbf{b}, \\ -\kappa\operatorname{div}\nabla T + i\omega\rho_0 C_V T + \frac{(\gamma-1)C_V\rho_0}{\beta}\operatorname{div}\mathbf{V} &= \rho_0 h, \\ p - \frac{(\gamma-1)C_V\rho_0}{\beta T_0}(T + \frac{i}{\omega\beta}\operatorname{div}\mathbf{V}) &= 0. \end{aligned} \tag{7}$$

Suitable boundary conditions for the velocity and temperature have to be prescribed.

- We assume that the velocity boundary condition in the normal direction to the boundary is given by

$$\mathbf{V} \cdot \mathbf{e}_n = \overline{V}_n \tag{8}$$

where $\overline{V}_n$ is the given value of the normal velocity. On the complement of the subset where (8) is imposed the surface force boundary condition in the normal direction to the boundary is defined by requiring that either

$$\mathbf{s}(\mathbf{e}_n) \cdot \mathbf{e}_n = \overline{s}_n \tag{9}$$

or

$$\mathbf{s}(\mathbf{e}_n) \cdot \mathbf{e}_n = Z_V(\mathbf{V} \cdot \mathbf{e}_n). \tag{10}$$

Here $\mathbf{s}(\mathbf{e}_n) = \mathbf{T}\mathbf{e}_n$ is the surface force vector, $\overline{s}_n$ is the given value of the normal surface force and Z_V is a complex quantity referred to as the specific acoustic impedance.

- The boundary conditions in the tangential directions to the boundary are given on pairs of complementary subsets of the boundary by

$$\mathbf{V}\cdot\mathbf{e}_s = \overline{V}_s \quad \text{or} \quad \mathbf{s}(\mathbf{e}_n)\cdot\mathbf{e}_s = \overline{s}_s \tag{11}$$

and

$$\mathbf{V}\cdot\mathbf{e}_t = \overline{V}_t \quad \text{or} \quad \mathbf{s}(\mathbf{e}_n)\cdot\mathbf{e}_t = \overline{s}_t. \tag{12}$$

- Slip boundary conditions may also be defined.

- Finally, the boundary conditions for the temperature are assumed to be given by either

$$T = 0 \quad \text{or} \quad \frac{\partial T}{\partial n} = Z_T T, \tag{13}$$

where the complex quantity Z_T is referred to as the specific thermal impedance.

1.2 Further approximations

- More approximate models may be obtained easily from the full model by letting some of the physical parameters of the model to vanish and relaxing the boundary conditions suitably.

- It is noted that in the case of vanishing heat conductivity (no boundary conditions associated with the temperature) the pressure satisfies the so-called viscous wave equation:

$$[1+\frac{i\omega(\lambda+2\mu)}{\rho_0 c^2}]\Delta p+\frac{\omega^2}{c^2}p=g \tag{14}$$

  where c is the adiabatic sound speed and g is known.

- In the case of vanishing heat conductivity and viscosity the pressure satisfies the Helmholtz equation:

$$\Delta p+\frac{\omega^2}{c^2}p=g. \tag{15}$$

- In this case no boundary conditions are associated with the temperature and the velocity (or surface traction) can be prescribed only in the normal direction.

- There are separate solvers for the Helmholtz equation available in Elmer.

2. Examples

The solution of the full equations is often challenging:

- The problem has a multi-scale character. The dissipative effects of viscosity and heat conduction are pronounced only in a thin boundary layer adjacent to a solid boundary. The boundary layer thickness is much smaller than the wavelength of the sound. The ratio of the thickness of the viscous boundary layer to the wavelength takes values $6.4 \cdot 10^{-4} \ldots 2.0 \cdot 10^{-5}$. Also the thickness of the thermal boundary layer is of a comparable size.

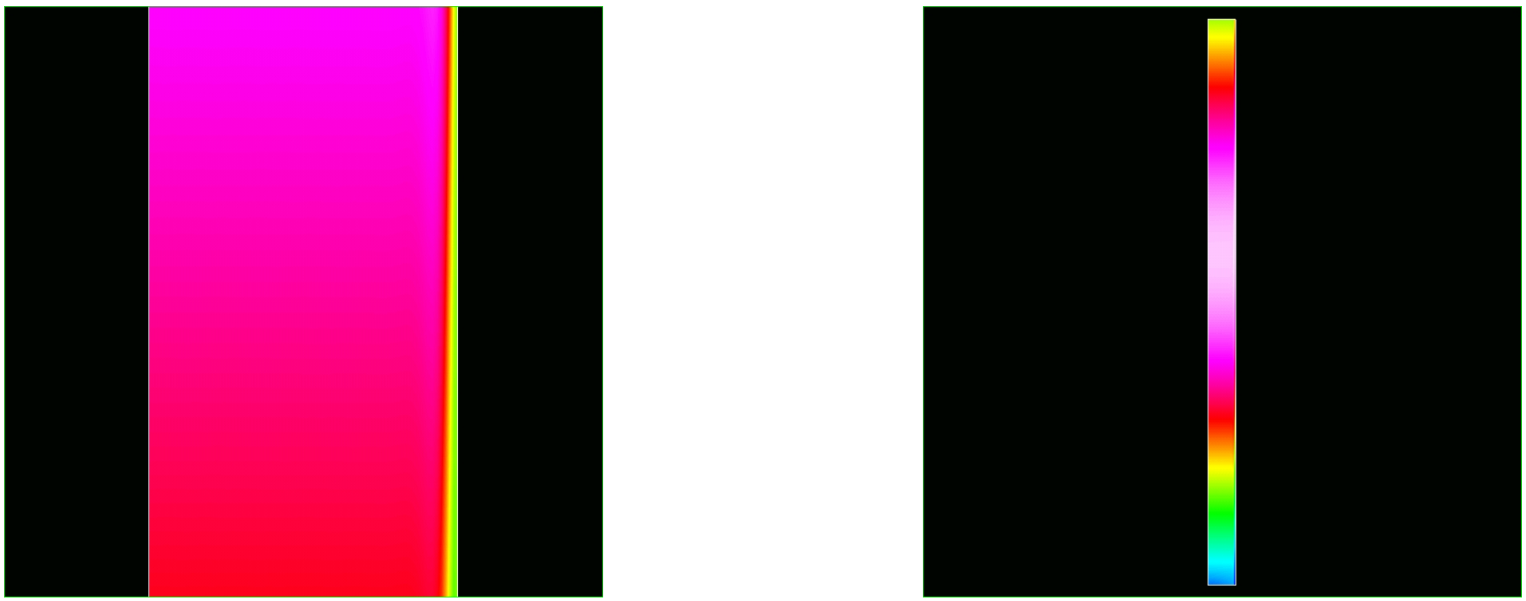

Figure 1: A fluctuation of temperature in a straight tube.

- Computational domains are often complicated. The construction of proper finite element meshes is not straightforward owing to the presence of boundary layers

- Owing to complicated geometries and the presence of rapidly decaying solution components, the linear systems resulting from the discretization can have a very large order.

  Can thermal and viscous effects be important?

- To demonstrate the importance of including dissipative effects we consider a simple test problem shown in Figure 2. We consider frequencies which are close to a resonance frequency of the system.

- We compute the real and imaginary parts of the acoustic impedance

$$Z = \frac{1}{A} \frac{\int_A p \, dA}{\int_A \mathbf{V} \cdot \mathbf{e}_n \, dA}$$

  where A is the surface on which the wave source is defined.

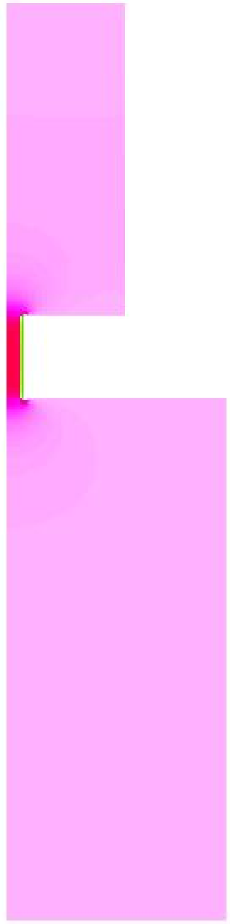

Figure 2: The geometry for the example problem. The tube contains air and the length of the tube is 4.4 cm.

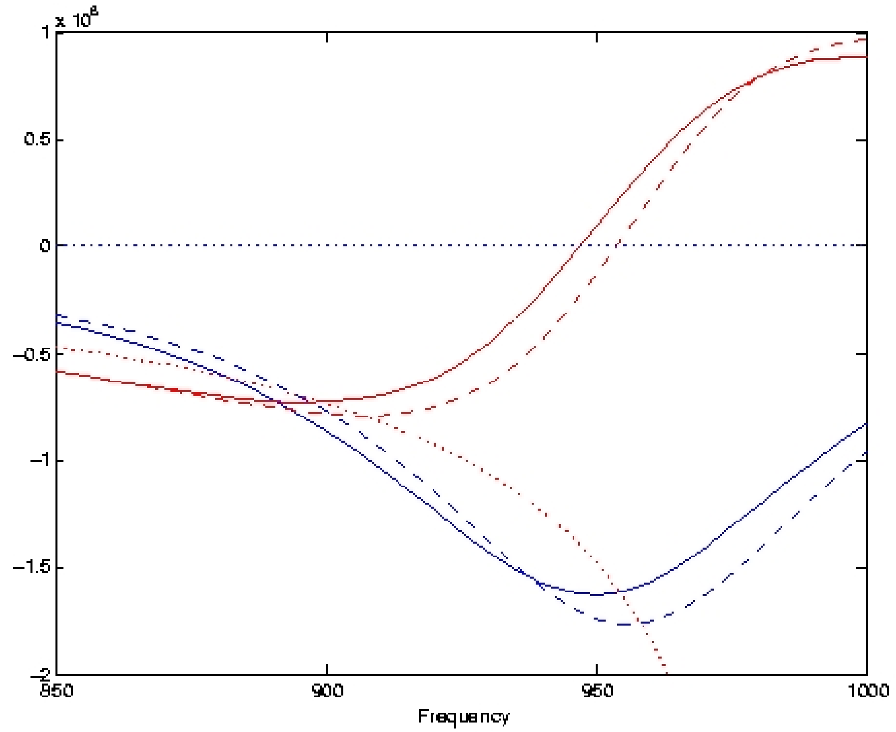

Figure 3: Different mathematical models of fluid flow may give completely unlike predictions for the response of an acoustic system. The blue and red curves are the real and imaginary parts of the impedance which are obtained by the full model (solid line), the model based on neglecting the effect of heat conduction (dashed line) and the model based on neglecting the effects of heat conduction and viscosity (dotted line).

- It is noted that an acoustic simulation over a range of frequencies can be done easily using the control structures available in ElmerSolver.

- The computation of special quantities (like impedances) can also be automated easily.

- Coupled analyses are also possible.

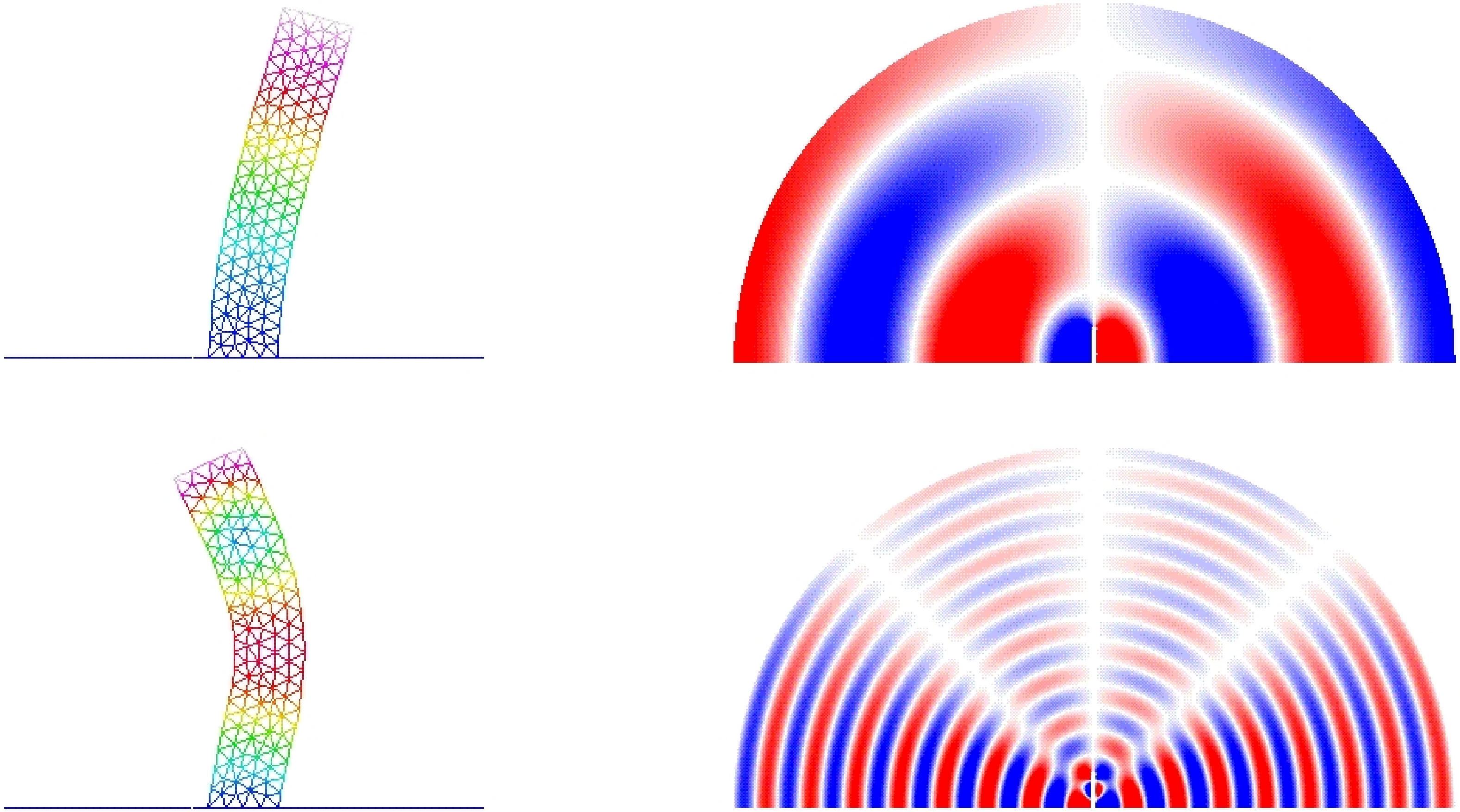

Figure 4: The sound pressure generated by a vibrating elastic beam. The natural vibration frequencies and mode shapes of the beam were computed using the linearized elasticity equations. The results were then used as input data for the Helmholtz equation which characterizes the propagation of pressure waves.

3. Concluding remarks

- One of the main challenges in the solution of the full equations is related to the solution of large linear systems which result form the discretization.

- The unknown fields can be strongly coupled which makes the development of efficient solution methods challenging. Our recent focus has been on developing preconditioned solution methods that utilize decoupled solution strategies. The idea in the preconditioning is to compute corrections to current velocity, temperature and pressure solutions in a decoupled manner.

- We are currently exploring an additional technique to improve the efficiency. Since the dissipative effects of viscosity and heat conduction are pronounced only in a thin boundary layer, outside this region the propagation of sound can be described using a simpler mathematical model (the Helmholtz equation).

- The idea is thus to solve different mathematical models of fluid flow on different regions of space. Suitable interface conditions for the two systems of equations have been developed.

- The strong coupling can also be problematic in the case of fluid-structure interaction problems (in particular when structural components are thin). We have been developing strongly coupled solution algorithms for such problems.

The end of the presentation

The first page