

ELMER, a computational tool for PDEs - Application to vibroacoustics

Juha Ruokolainen and Mikko Lyly
CSC - Scientific Computing Ltd.

ELMER is a software package for solving Partial Differential Equations (PDEs) by Finite Element Method (FEM). ELMER has been developed at CSC in collaboration with Technical Research Center of Finland (VTT), Helsinki University of Technology and University of Jyväskylä.

In this article we present a case study of a new feature included in ELMER, vibroacoustics. Vibroacoustic problem consists of solving the natural vibration modes of elastic bodies, and the pressure waves generated by the vibration.

Examples of vibrating elastic bodies are a guitar string or a tuning fork; you can probably come up with better examples yourself. The computational domain for the pressure waves is usually the air surrounding the elastic body. Other solid bodies, if present, will scatter the pressure waves.

We will go through an example step by step. We will first present the mathematical model, create the element mesh, define material parameters and boundary conditions and finally solve the system and visualize the results.

The mathematical model for vibroacoustic problems

The linearized dynamical equations for elastic bodies, without external forces can be written as

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla \cdot \boldsymbol{\tau}(\mathbf{u}) = 0, \quad (1)$$

where \mathbf{u} is the displacement vector and $\boldsymbol{\tau}$ is the *stress tensor*,

$$\boldsymbol{\tau}(\mathbf{u}) = \lambda(\nabla \cdot \mathbf{u})\mathbf{I} + \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \quad (2)$$

Here λ and μ are the first and second Lamé parameters respectively. The Lamé parameters are directly related to the more commonly available material parameters elastic modulus E and Poisson ratio σ :

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)}. \quad (3)$$

Fourier transform of the elasticity equations with respect to time variable gives

$$-\omega^2 \rho \mathbf{U} - \nabla \cdot \boldsymbol{\tau}(\mathbf{U}) = 0, \quad (4)$$

which, by standard finite element methodology, may be written in matrix form as

$$K\mathbf{U} = \omega^2 M\mathbf{U}, \quad (5)$$

where the K and M are the stiffness and mass matrices respectively. This equation may be considered as a generalized eigenvalue problem, giving the natural vibration modes of the elastic body.

The wave equation

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \nabla^2 p, \quad (6)$$

describes pressure waves in the air. The pressure and density variations and air speed are considered so small that dissipative effects and convective transport of momentum may be neglected. The parameter c is the speed of sound. The Fourier transform of the wave equation is called the *Helmholtz equation*:

$$-k^2 P - \nabla^2 P = 0, \quad (7)$$

where the wave number k is given by $k = \omega/c$. With a linear material law the whole equation is linear and no dispersion of harmonic components will occur. It is then possible to compute the response to any signal, for example the response to vibration of an elastic body, one Fourier component at a time.

To complete the system we need boundary conditions. In our example the elastic body is a tuning fork fixed at the bottom of the grip. For the Helmholtz equation we will need three kinds of boundary conditions. The first boundary condition is at the boundary between the vibrating body and the air, where we have to specify acceleration of the body:

$$\nabla P \cdot \mathbf{n} = \rho \omega^2 \mathbf{U} \cdot \mathbf{n}, \quad (8)$$

where ρ is the density of air. The second boundary condition is at the boundary between the solid bodies and the air:

$$\nabla P \cdot \mathbf{n} = 0, \quad (9)$$

describing reflection of the pressure waves from solid walls. The last boundary condition we will use is the so called Sommerfeldt or far field boundary condition, which approximates the continuation of the physical domain from where we have to cut the computational mesh:

$$\nabla P \cdot \mathbf{n} = \pm ikP. \quad (10)$$

In time domain this equation is

$$-\nabla p \cdot \mathbf{n} = \pm \frac{1}{c} \frac{\partial p}{\partial t}. \quad (11)$$

The sign of the time derivative term on the right represents incoming and outgoing waves respectively.

Setting up the computation: mesh generation, material parameters and boundary conditions

The ELMER package has a graphical interface called ELMER Front. This program is used as a tool to describe the computational problem. Defining the geometry, generating the computational mesh, selecting the equations to be solved, setting the material parameters and boundary conditions may all be done through ELMER Front.

ELMER Front has its own simple 2D geometry input format which will be used in this example. In addition, ELMER Front can read some of the most common geometry formats exported by various CAD programs. Even pre-defined meshes may be imported to ELMER Front for boundary condition settings and conversion to ELMER Solver format.

The geometry and the computational mesh of our example are shown in figure 1. Figure 2 shows the ELMER Front panel where we define mesh parameters, such as mesh density and element types. For the elasticity equations it is always a good idea to use higher order elements than linear, and so we have selected quadratic elements.

Figure 3 shows how to set boundary conditions, in this case the boundary conditions for fixing the bottom of the vibrating tuning fork.

The Solution of the equations

ELMER Solver discretizes the equations by Finite Element Method, and solves the resulting linear or nonlinear systems, providing input for further analysis and visualization of the results. In addition to the equations used in this example, ELMER Solver may solve several other sets of equations: the Navier-Stokes equations in both compressible and incompressible forms, magnetic induction equation, heat equation including diffuse gray radiation,

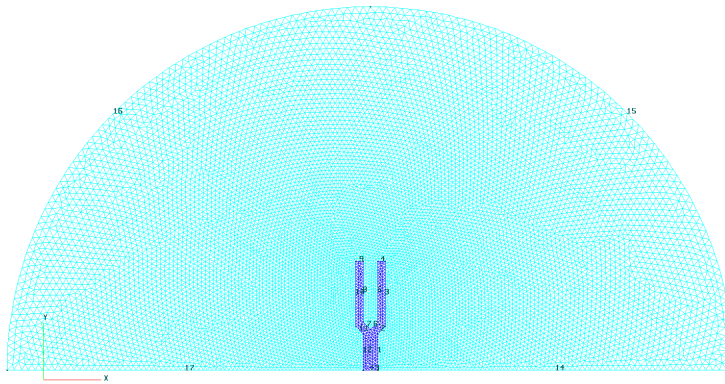


Figure 1: Mesh and geometry of the tuning fork and the air.

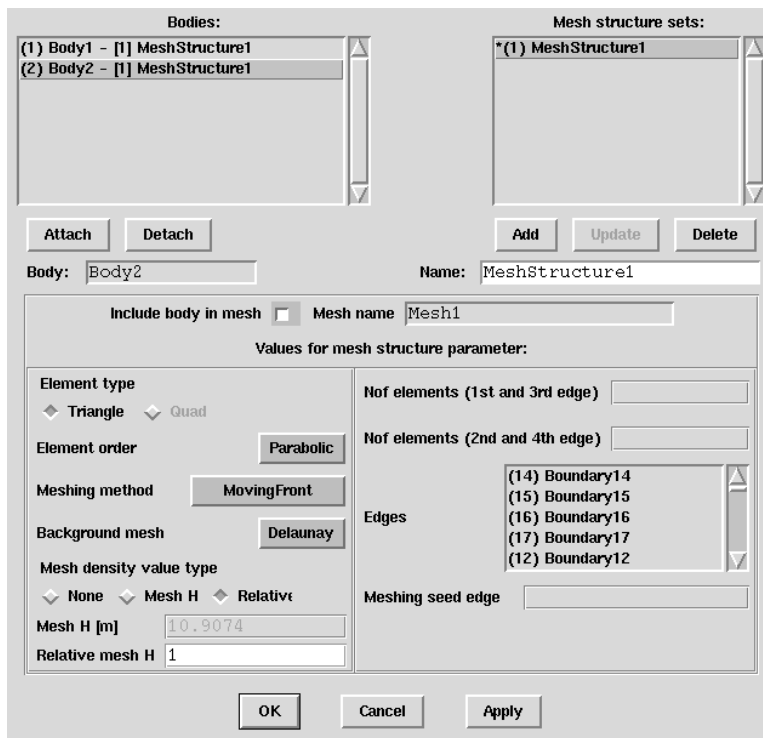


Figure 2: Mesh parameter setting panel of ELMER Front.

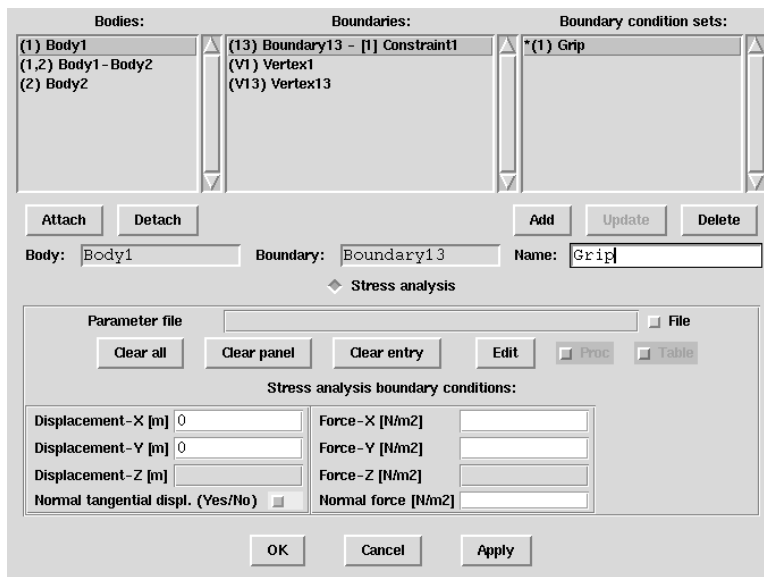


Figure 3: *Boundary condition setting panel of ELMER Front.*

and nonlinear elasticity and wave equations also in the time domain. A general advection-diffusion-reaction equation for scalar variables is also available. Any combination of these equations with various interdependencies may be solved at the same time. The solver is also easily extensible, so that user may define his/her own sets of equations.

Here we will compute the second lowest natural vibration mode of the tuning fork, and use it as input for the Helmholtz equation solver.

In this example, the linear equation system from the Helmholtz equation was solved by iterative solution method (BiCGStab) using incomplete LU factorization (ILU(T)) as a preconditioner.

The generalized eigenvalue problem, for the structural vibration modes, was solved using the ARPACK package developed at the Department of Computational and Applied Mathematics at Rice University, Texas.

Visualizing the results

ELMER Solver output is a large text file of tabularized numbers. In this case the numbers are the vibration amplitudes of material points of the tuning fork, and coefficients of the Fourier components of pressure for each of the natural frequencies.

ELMER Post may be used to display the results in a more understandable form. ELMER Post can display, for example, the deformed geometry of the

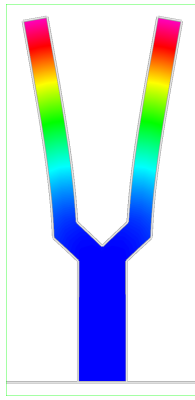


Figure 4: *The vibration mode shape of the tuning fork.*

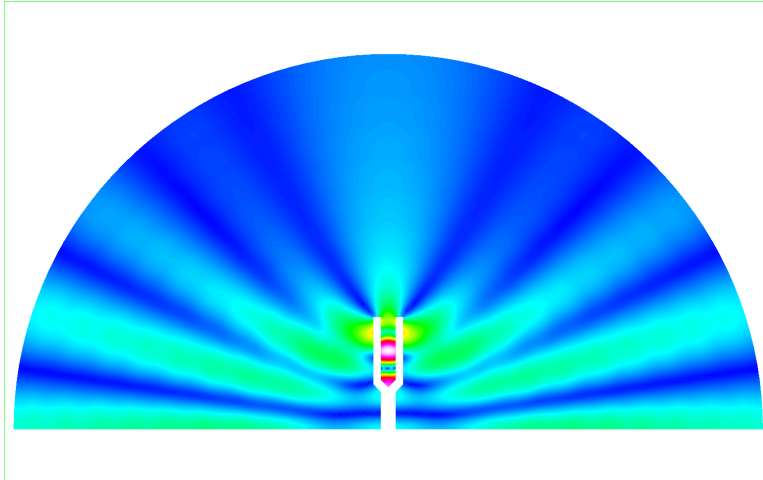


Figure 5: *Energy density of the pressure field.*

elastic body. It may also be used to compute the time variation of the pressure field,

$$P(t) = \text{Re}(P)\cos(\omega t) \pm \text{Im}(P)\sin(\omega t), \quad (12)$$

and display the time evolution as an animation. The vibration of the elastic body may be displayed in the same manner.

Figure 4 shows the vibration mode shape of the tuning fork. Figure 5 is the energy density contained in the pressure field at this particular frequency. Figure 6 shows the pressure field at one instance.

Additional information

The ELMER package is available from CSC; for more information contact

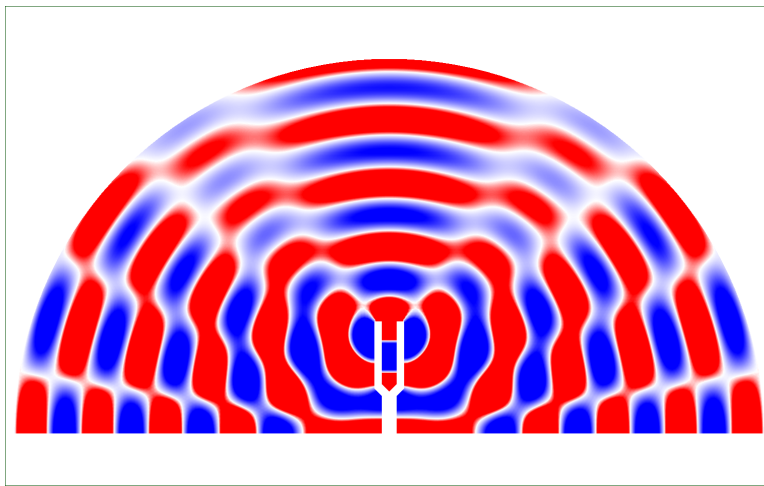


Figure 6: *The pressure field.*

the authors. For academic use the package is provided without charge.

Also check out our web pages from:

<http://www.csc.fi/elmer/>

Information about ARPACK is available from

<http://www.caam.rice.edu/software/ARPACK>