European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS 2004
P. Neittaanmäki, T. Rossi, K. Majava, and O. Pironneau (eds.)
J. Périaux and M. Křížek (assoc. eds.)
Jyväskylä, 24–28 July 2004

A FINITE ELEMENT METHOD FOR THE MODELING OF THERMO-VISCOUS EFFECTS IN ACOUSTICS

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Key words: acoustics, finite elements, dissipation, resonance

Abstract. *A finite element approach to the solution of acoustic field equations including the effects of viscosity and heat conduction is presented. The mathematical model considered is based on the linearized Navier-Stokes, continuity and energy equations. The solution to the field equations is assumed to be time-harmonic, so that what remains to be solved is the spatial dependence of the amplitudes of primary unknowns. The original set of field equations may be reduced to a system with the amplitudes of the velocities and temperature as the only unknown fields. This reduced system has often been used as a starting point in the analytical treatment and could also be thought to be a natural starting point for the finite element formulation. However, special care has to be taken in the finite element discretization of the reduced system. Firstly, a stable pair of finite element spaces for the velocities and temperature has to be chosen. Secondly, we show by writing the system in a dimensionless form that in certain cases, commonly met in practice, an additional difficulty known to lead to a finite element locking effect is expected. In fact, the same problem is generally met in the finite element analysis of nearly incompressible flows. To avoid unnecessary error amplification we rewrite the system as a mixed problem with a pressure-like quantity as auxiliary unknown. The resulting formulation is discretized using an enhanced MINI finite element. The reliability and applicability of the resulting finite element method is demonstrated by numerical experiments.*

1 INTRODUCTION

This work is concerned with the development of finite element methods for the approximation of acoustic field equations which include the effects of viscosity and heat conduction. The mathematical model considered is based on the linearized Navier-Stokes, continuity and energy equations supplemented by suitable equations of state. We confine our consideration to the time-harmonic analysis.

The approach we take here has not customarily been used in the computational acoustics, as there have been many attempts to further simplify the system of field equations and use simplified models as a starting point for the development of numerical methods (for an overview of models see for example [1]). Such alternative approaches are partly motivated by the fact that the effects of viscosity and heat conduction are pronounced only in a thin boundary layer near a solid boundary. This makes it possible to split the solution to the field equations into the propagational mode and the edge effect characterized by boundary-layer equations. By imposing appropriate boundary conditions, the complete solution may be obtained in terms of these two modes [2].

In the condition of resonance, the question of how to obtain the complete solution using solutions to simpler models becomes far from trivial, as the basic wave equation characterizing the propagational mode is then nearly singular. This difficulty in part motivates our direct starting from the full system of acoustic field equations.

The paper is organised as follows. The mathematical model we consider is described in Section 2. In Section 3 we make an observation which indicates that special care has to be taken in the finite element discretization of the field equations and leads us to propose a nonstandard finite element method for the approximation of the model. Numerical experiments which demonstrate the reliability and applicability of the proposed method are provided in Section 4.

2 FIELD EQUATIONS

We consider the solution of acoustic equations based on the linearized Navier-Stokes, continuity and energy equations

$$
\begin{aligned}
\rho_0 \frac{\partial \mathbf{v}}{\partial t} &= -\operatorname{grad} p + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{v} + \mu \operatorname{div} \operatorname{grad} \mathbf{v} + \rho_0 \mathbf{b}, \\
\frac{\partial \rho}{\partial t} &= -\rho_0 \operatorname{div} \mathbf{v}, \\
\rho_0 \frac{du}{dt} &= \kappa \operatorname{div} \operatorname{grad} T - p_0 \operatorname{div} \mathbf{v} + \rho_0 h,
\end{aligned}
\tag{1}
$$

where $\mathbf{v}$, ρ, p and T are the velocity, density, pressure and temperature, $\mathbf{b}$ is the body force (per unit mass), λ and μ are parameters characterizing the viscosity of the fluid, u is the specific internal energy, κ is the heat conductivity and h is the internal supply of heat. In addition, the notations ρ_0, p_0 and T_0 are here used for the values of ρ, p and T

at the equilibrium state. The equations (1) are supplemented by the assumption that the fluid obeys the ideal gas law

$$\frac{\partial p}{\partial t} = R(\rho_0 \frac{\partial T}{\partial t} + T_0 \frac{\partial \rho}{\partial t}), \tag{2}$$

and the further assumption that the specific internal energy u is a function only of the temperature implying that

$$\frac{du}{dt} = C_V T_0 (\frac{1}{p_0}\frac{\partial p}{\partial t} - \frac{1}{\rho_0}\frac{\partial \rho}{\partial t}). \tag{3}$$

Here one has $R = (\gamma - 1)C_V$ with C_V and γ being the specific heat at constant volume (per unit mass) and the ratio of the specific heats at constant pressure and constant volume.

Confining our consideration to time-harmonic analysis, we assume that the solution of the system (1)–(3) is of the form

$$\begin{aligned} \mathbf{v}(\mathbf{x}, t) &= \mathbf{v}(\mathbf{x}) \exp(i\omega t), \\ \rho(\mathbf{x}, t) &= \rho_0 + \rho(\mathbf{x}) \exp(i\omega t), \\ p(\mathbf{x}, t) &= p_0 + p(\mathbf{x}) \exp(i\omega t), \\ T(\mathbf{x}, t) &= T_0 + T(\mathbf{x}) \exp(i\omega t), \end{aligned} \tag{4}$$

where ω is the angular frequency. By the substitution of (4), the equations (1)–(3) may be reduced to a system with the amplitudes of the disturbances of the velocity and temperature $\mathbf{v}(\mathbf{x})$ and $T(\mathbf{x})$ as the only unknown fields. The reduced system may be written as

$$\begin{aligned} i\omega\rho_0 \mathbf{v} &= -\frac{p_0}{T_0} \operatorname{grad} T + (\lambda + \mu - \frac{ip_0}{\omega}) \operatorname{grad} \operatorname{div} \mathbf{v} + \mu \operatorname{div} \operatorname{grad} \mathbf{v} + \rho_0 \mathbf{b}, \\ \kappa \operatorname{div} \operatorname{grad} T &= i\omega\rho_0 C_V T + \rho_0 T_0 C_V (\gamma - 1) \operatorname{div} \mathbf{v} - \rho_0 h, \end{aligned} \tag{5}$$

and the disturbances of the pressure and density fields $p(\mathbf{x})$ and $\rho(\mathbf{x})$ can be obtained in terms of $\mathbf{v}$ and T from the relations

$$p = \frac{p_0}{T_0} T + \frac{ip_0}{\omega} \operatorname{div} \mathbf{v} \quad \text{and} \quad \rho = \frac{i\rho_0}{\omega} \operatorname{div} \mathbf{v}. \tag{6}$$

We assume that the system (5) is supplemented by the boundary conditions that are of the form

$$\mathbf{v} = \mathbf{g} \quad \text{and} \quad T = 0 \tag{7}$$

on the boundary $\partial\Omega$ of the computational domain $\Omega \subset \mathbb{R}^n$. More general boundary conditions could also be imposed without difficulty.

It will be convenient to transform the field equations (5) into a dimensionless form. To this end we introduce the dimensionless quantities

$$\theta = kL \qquad \epsilon = \frac{\mu}{\rho_0 \omega L^2} \qquad Pr = \frac{\gamma C_V \mu}{\kappa} \tag{8}$$

where $L = \mathrm{diam}(\Omega)$ and $k = \omega/c$ with the adiabatic sound speed c being defined by the relation $\rho_0 c^2 = \gamma p_0$. It is noted that the dimensionless parameter θ characterizes the ratio of the characteristic length L to the acoustic wavelength, while the square root of ϵ characterizes the ratio of the thickness of viscous boundary layer to L. Similarly, the square root of the parameter Pr, known as the Prandtl number, characterizes the ratio of the thickness of viscous boundary layer to that of thermal boundary layer.

We also introduce dimensionless unknowns by setting

$$\mathbf{v}' = \frac{\mathbf{v}}{c} \qquad p' = \frac{p}{p_0} \qquad T' = \frac{T}{\gamma \theta T_0} \tag{9}$$

and define scaled coordinates x_i', $i = 1, ..., n$, by $x_i' = L^{-1} x_i$. The computational domain in the scaled coordinates will be denoted by Ω'.

By the substitution of the dimensionless quantities and the replacement of derivatives by ones with respect to the dimensionless coordinates using the transformation $\partial/\partial x_i = (1/L)\partial/\partial x_i'$, the field equations (5) in the absence of $\mathbf{b}$ and h may be reduced to

$$\begin{aligned} i\mathbf{v}' &= -\nabla T' - \frac{i}{\gamma\theta^2} \nabla \operatorname{Div} \mathbf{v}' + (\eta + 1)\epsilon \nabla \operatorname{Div} \mathbf{v}' + \epsilon \Delta \mathbf{v}', \\ \frac{\epsilon(\gamma\theta)^2}{(\gamma - 1)Pr} \Delta T' &= \frac{i\gamma\theta^2}{\gamma - 1} T' + \operatorname{Div} \mathbf{v}', \end{aligned} \tag{10}$$

where ∇ and Div are the gradient and divergence with respect to the dimensionless coordinates, $\Delta = \operatorname{Div} \nabla$ and

$$\eta = \lambda/\mu. \tag{11}$$

3 FINITE ELEMENT FORMULATION

The reduced system (5) has often been used as a starting point in the analytical treatment and could also be thought to be a natural starting point for the finite element formulation. However, special care has to be taken in the finite element approximation of this system, as the straightforward discretization, say using a same finite element space for the approximation of the velocities and temperature, may not lead to a reliable numerical method.

The nature of difficulties arising in the finite element approximation becomes apparent when a simple order-of-magnitude analysis of the field equations written in the dimensionless form is done. To motivate the use of a nonstandard finite element formulation presented in this paper, we begin by giving such analysis in Section 3.1. The finite element method we propose is described in Section 3.2.

3.1 Order-of-magnitude analysis

To demonstrate the nature of the difficulties arising in the finite element approximation, let us consider the scaled field equations (10) and split the solution into the edge effect solution which decays exponentially as we go away from the boundary and the remaining solution component corresponding to the propagational wave. We consider a case for which $\epsilon \ll 1$ so that the thickness of the zone in which the viscous boundary layer has a significant effect is relatively thin as compared with the length L. In addition, we assume that $\theta \ll 1$ so that the propagational wave varies smoothly in the length scale L. The subsequent differentiations of the smooth solution component do not thus lead to a change in the order of magnitude, i.e. we have for any integer α and $i = 1, ..., n$

$$\frac{\partial^\alpha \mathbf{v}'_s}{\partial x_i'^\alpha} = O(\mathbf{v}'_s) \quad \text{and} \quad \frac{\partial^\alpha T'_s}{\partial x_i'^\alpha} = O(T'_s) \tag{12}$$

with $\mathbf{v}'_s$ and T'_s being the smooth components of the scaled velocity and temperature.

If the properties of the medium are such that $\mathrm{Pr} = O(1)$ and $\eta = O(1)$, the field equations characterizing the smooth solution component may be reduced to

$$\begin{aligned} i\mathbf{v}'_s &= -\nabla T'_s - \frac{i}{\gamma\theta^2} \nabla \operatorname{Div} \mathbf{v}'_s + O(\epsilon \mathbf{v}'_s), \\ \frac{i\gamma\theta^2}{\gamma - 1} T'_s &+ \operatorname{Div} \mathbf{v}'_s = O(\epsilon\theta^2 T'_s). \end{aligned} \tag{13}$$

In view of (12) and (13) the velocity field $\mathbf{v}'_s$ satisfies

$$\operatorname{Div} \mathbf{v}'_s \sim -\theta^2 \mathbf{v}'_s,$$

which implies that outside the boundary layer zone the velocity field $\mathbf{v}'$ is nearly incompressible when θ is small.

It is known from the finite element theory that the finite element spaces for the approximation of the velocity and temperature in the basic variational formulation of (13) must be chosen carefully. For example the use of a same low-order approximation for each of the unknown fields may not lead to reliable numerical method if θ is small enough. Even with having a stable pair of finite element spaces for the velocity and temperature, we still have an other difficulty owing to the presence of the penalty-like term

$$\frac{i}{\gamma\theta^2} \nabla \operatorname{Div} \mathbf{v}'_s$$

in the first equation of the system (13). If $\theta \ll 1$, the standard finite element approximation of this term is known to lead to a parametric error-growth phenomenon referred to as the locking effect [3]. It is noted that a similar effect is also expected if, instead of having $\lambda \sim \mu$, we let $\lambda \longrightarrow \infty$.

The above order-of-magnitude analysis indicates that the finite element approximation of the smooth solution component of the field equations (10) (or equivalently (5)) is a nontrivial task. To avoid parametric error-growth effects we will formulate the problem as a mixed problem.

3.2 Finite element discretization

To present the finite element formulation we continue to work with the scaled field equations. The solution to the original variables can be recovered from this formulation simply by rescaling the dimensionless solution.

In order to avoid parametric error-growth effects we introduce an auxiliary unknown ϕ' by rewriting the system (10) as

$$\begin{aligned} i\mathbf{v}' &= -\nabla T' - \nabla \phi' + \epsilon \nabla \operatorname{Div} \mathbf{v}' + \epsilon \Delta \mathbf{v}', \\ \frac{\epsilon(\gamma\theta)^2}{(\gamma-1)Pr}\Delta T' &= \frac{i\gamma\theta^2}{\gamma-1}T' + \frac{\gamma\theta^2}{i-\gamma\theta^2\eta\epsilon}\phi', \\ \frac{\gamma\theta^2}{i-\gamma\theta^2\eta\epsilon}\phi' &= \operatorname{Div}\mathbf{v}'. \end{aligned} \tag{14}$$

By setting

$$\begin{aligned} &V = \{\mathbf{u} \in [H^1(\Omega')]^n \mid \mathbf{u} = (1/c)\mathbf{g} \text{ on } \partial\Omega'\}, \quad W = [H^1_0(\Omega')]^n, \\ &X = H^1_0(\Omega'), \quad Y = L^2(\Omega') \end{aligned} \tag{15}$$

(here the spaces $L^2(\Omega')$, $H^1(\Omega')$ and $H^1_0(\Omega')$ are defined in the usual way [3]), the variational formulation of the system (14) supplemented by the boundary conditions that are equivalent to (7) may be written in the following form: Find $(\mathbf{v}', T', \phi') \in V \times X \times Y$ such that

$$\begin{aligned} a(\mathbf{v}', \mathbf{w}') + b(\overline{\mathbf{w}}', \overline{T}') + b(\overline{\mathbf{w}}', \overline{\phi}') &= 0, \\ c(T', q') + d(\phi', q') &= 0, \\ b(\mathbf{v}', \varphi') + d(\phi', \varphi') &= 0 \end{aligned} \tag{16}$$

for all $(\mathbf{w}', q', \varphi') \in W \times X \times Y$ with the sesquilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$ and $d(\cdot,\cdot)$ being defined by

$$\begin{aligned} a(\mathbf{v}, \mathbf{w}) &= i\int_{\Omega'} \mathbf{v}\cdot\overline{\mathbf{w}}\,\mathrm{d}\Omega' + \epsilon\int_{\Omega'} \operatorname{Div}\mathbf{v}\operatorname{Div}\overline{\mathbf{w}}\,\mathrm{d}\Omega' + \epsilon\int_{\Omega'} \nabla\mathbf{v}\cdot\nabla\overline{\mathbf{w}}\,\mathrm{d}\Omega', \\ b(\mathbf{v}, q) &= -\int_{\Omega'} \operatorname{Div}\mathbf{v}\overline{q}\,\mathrm{d}\Omega', \\ c(T, q) &= \frac{\epsilon(\gamma\theta)^2}{(\gamma-1)Pr}\int_{\Omega'} \nabla T\cdot\nabla\overline{q}\,\mathrm{d}\Omega' + \frac{i\gamma\theta^2}{\gamma-1}\int_{\Omega'} T\overline{q}\,\mathrm{d}\Omega', \\ d(\phi, \varphi) &= \frac{\gamma\theta^2}{i-\gamma\theta^2\eta\epsilon}\int_{\Omega'} \phi\overline{\varphi}\,\mathrm{d}\Omega'. \end{aligned} \tag{17}$$

It is noted that the pressure may now be obtained without differentiation from the equation

$$p = \gamma\theta p_0(T' + \frac{i}{i - \gamma\theta^2\eta\epsilon}\phi'). \tag{18}$$

We define the finite element discretization of the variational problem by choosing finite element spaces $V_h \subset [H^1(\Omega')]^n$, $X_h \subset X$ and $Y_h \subset Y$ for the approximation of the velocity, temperature and the pressure-like quantity ϕ' and solving the finite element solution from the following problem: Find $(\mathbf{v}'_h, T'_h, \phi'_h) \in (V_h \cap V) \times X_h \times Y_h$ such that

$$\begin{aligned} a(\mathbf{v}'_h, \mathbf{w}') + b(\overline{\mathbf{w}}', \overline{T}'_h) + b(\overline{\mathbf{w}}', \overline{\phi}'_h) &= 0, \\ c(T'_h, q') + d(\phi'_h, q') &= 0, \\ b(\mathbf{v}'_h, \varphi') + d(\phi'_h, \varphi') &= 0 \end{aligned} \tag{19}$$

for all $(\mathbf{w}', q', \varphi') \in (V_h \cap W) \times X_h \times Y_h$.

Given partitioning $\mathcal{T}$ of the computational domain into elements K we define

$$\begin{aligned} &V_h = \{\mathbf{u} \mid \mathbf{u} \text{ is continuous and } \mathbf{u}_{|K} \in V(K)\ \forall K \in \mathcal{T}\}, \\ &X_h = \{q \mid q \text{ is continuous and } q_{|K} \in X(K)\ \forall K \in \mathcal{T}\}, \quad Y_h = X_h, \end{aligned} \tag{20}$$

where the local spaces $V(K)$ and $X(K)$ are constructed in the usual way for given spaces $V(\hat{K})$ and $X(\hat{K})$ on a reference element $\hat{K}$. In the case of triangles we take $X(\hat{K}) = P_k(\hat{K})$ where $P_k(\hat{K})$ is the space of polynomials at most degree k and $V(\hat{K}) = [P_k(\hat{K})]^n \oplus [B(\hat{K})]^n$ with $B(\hat{K}) = P_{k+n+1}(\hat{K}) \cap H^1_0(\hat{K})$.

In the case of quadrilaterals we take $X(\hat{K}) = Q_k(\hat{K})$ where $Q_k(\hat{K})$ is the space of polynomials at most degree k in each coordinate, while the velocity space is defined by $V(\hat{K}) = [Q_k(\hat{K})]^n \oplus [B(\hat{K})]^n$ with $B(\hat{K}) = Q_{k+n}(\hat{K}) \cap H^1_0(\hat{K})$. The considered family of methods can also be extended to three dimensions, but our current experience is limited mainly to cases in two dimensions.

It is noted that the degrees of freedom associated with the bubble space $B(K)$ may be eliminated using the static condensation, so we obtain an formulation that employs equal-order approximation for all the unknown fields. We also note that the method proposed is an enhancement of the finite element known as the MINI element in the context of the Stokes equation [3].

We conclude this section by noting that by the introduction of the auxiliary unknown ϕ', the variational formulation of the problem could be given in variety of forms. One alternative formulation would be to write the variational problem as: Find $(\mathbf{v}', T', \phi') \in V \times X \times Y$ such that

$$\begin{aligned} a(\mathbf{v}', \mathbf{w}') + b(\overline{\mathbf{w}}', \overline{T}') + b(\overline{\mathbf{w}}', \overline{\phi}') &= 0, \\ b(\mathbf{v}', q') - c(T', q') &= 0, \\ b(\mathbf{v}', \varphi') + d(\phi', \varphi') &= 0 \end{aligned} \tag{21}$$

for all $(\mathbf{w}', q', \varphi') \in W \times X \times Y$. We have found experimentally that the finite element approximation of (21) using the finite elements considered here does not lead to as stable method as one resulting from the approximation of (16). It appears that the explanation of these experimental observations would require careful mathematical analysis.

4 NUMERICAL EXPERIMENTS

The finite element method described above has been implemented in Elmer [4] which is finite element software for the modeling of multi-physical problems. To demonstrate the reliability of the proposed method with respect to variations of the parameter θ we consider three cases corresponding to different values of the frequency $f = \omega/(2\pi)$.

The computational domain is taken to be $\Omega = \{\mathbf{x} = (x_1, x_2) \mid \mathbf{x} \in (0, H_1) \times (0, L_1) \cup (0, H_2) \times [L_1, L_2] \cup (0, H_3) \times (L_2, L_3)\}$ with $H_1 = 9.25 \cdot 10^{-3}$, $L_1 = 2.5 \cdot 10^{-2}$, $H_2 = 7.5 \cdot 10^{-4}$, $L_2 = 2.9 \cdot 10^{-2}$, $H_3 = 5.0 \cdot 10^{-3}$ and $L_3 = 4.4 \cdot 10^{-2}$. We assume that the medium is air at standard conditions and take $T_0 = 293$, $p_0 = 1.013 \cdot 10^5$, $\rho_0 = 1.21$, $\mu = 1.82 \cdot 10^{-5}$, $\lambda = -2/3\mu$, $\kappa = 2.6 \cdot 10^{-2}$, $C_V = 717.17$ and $\gamma = 1.4$. Here all values are expressed in SI units.

The wave source is defined by imposing a non-vanishing velocity amplitude on the edge $x_2 = L_3$. We take $v_1(x_1) = 0$ and $v_2(x_1) = J_0(\alpha x_1)$ where J_0 is the Bessel function with the maximum value scaled to the value 0.001 and α is a scaling factor defined so that $x_1 = \alpha H_3$ is the first root of $J_0(x_1)$. In addition, we assume that the heat flux vanishes on this edge (the variational formulation given in the previous section was extended to cover cases involving heat flux and traction boundary conditions). On the edge $x_1 = 0$ the boundary conditions are deduced on the basis of symmetry. On the remaining part of the boundary the conventional wall boundary conditions are imposed, i.e. all the components of the velocity vector as well as the disturbances of the temperature are prescribed to vanish on these boundaries.

The finite element mesh used in the computations consisted of both linear triangles and bilinear quadrilaterals (i.e. we used elements of degree $k = 1$) with the total number of nodes being 6226. Special care was taken in the mesh generation so as to capture the boundary layer effects near the walls with adequate accuracy. The finite element mesh used in the computations is presented in Figure 1.

The problem was solved using the frequencies $f = 20, 200, 2000$. It is noted that the parameter θ takes accordingly values $\theta \approx 0.016, 0.16, 1.6$ with L being defined to be $L = L_3$. The profiles of the temperature T and pressure p at the end $x_2 = L_3$ are presented in Figures 2–4. We note that regardless of the value of f the solution profiles are smooth and no spurious oscillations are detected. These results indicate that the performance of the method does not depend on the parameter θ. More numerical results may be found in [5] where some experimental verification of the solution methodology is also given.

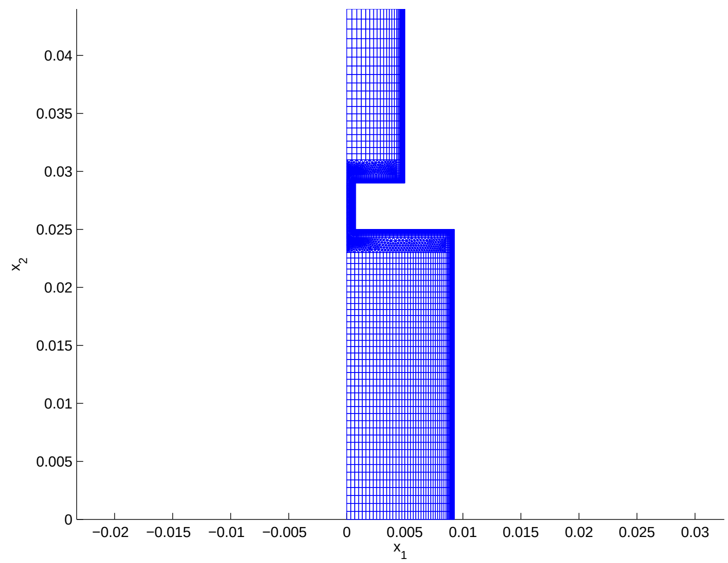

Figure 1: The finite element mesh used in the computations.

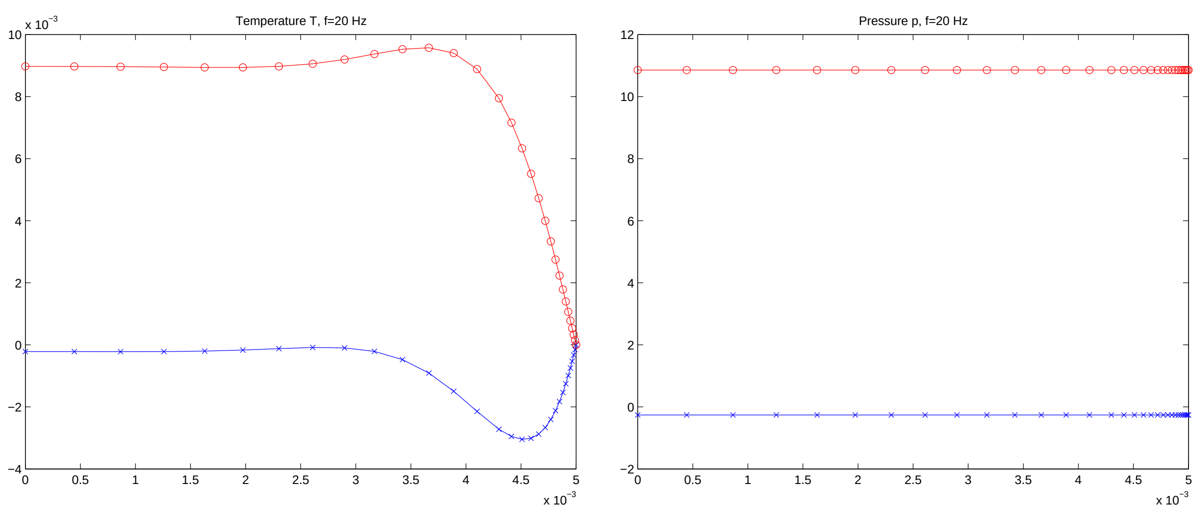

Figure 2: The real and imaginary parts of the temperature T and pressure p at the end $x_2 = L_3$ for $f = 20$ Hz. The values of the real and imaginary parts at the nodes are displayed by $\times$ and $\circ$, respectively.

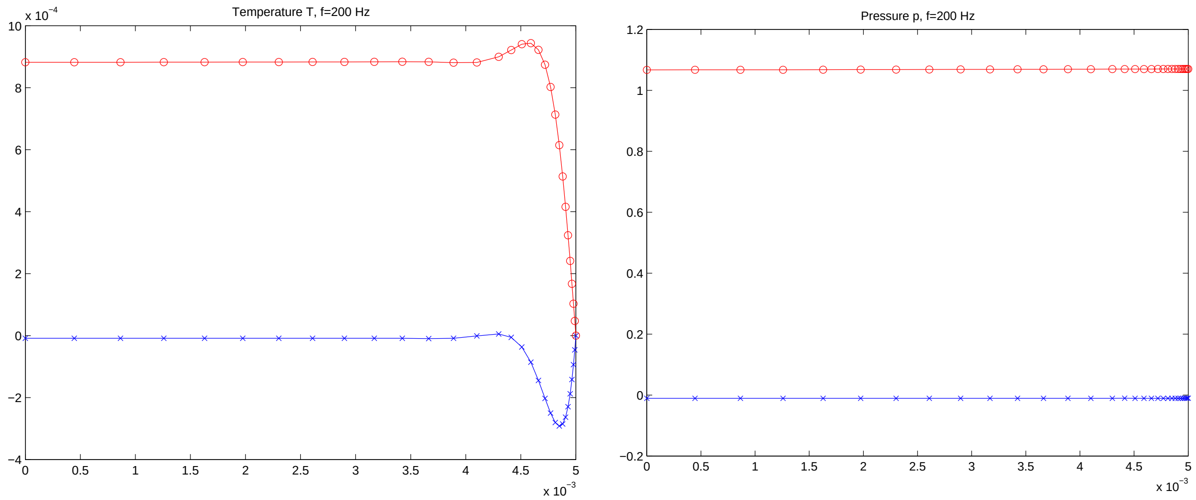

Figure 3: The real and imaginary parts of the temperature T and pressure p at the end $x_2 = L_3$ for $f = 200$ Hz. The values of the real and imaginary parts at the nodes are displayed by $\times$ and $\circ$, respectively.

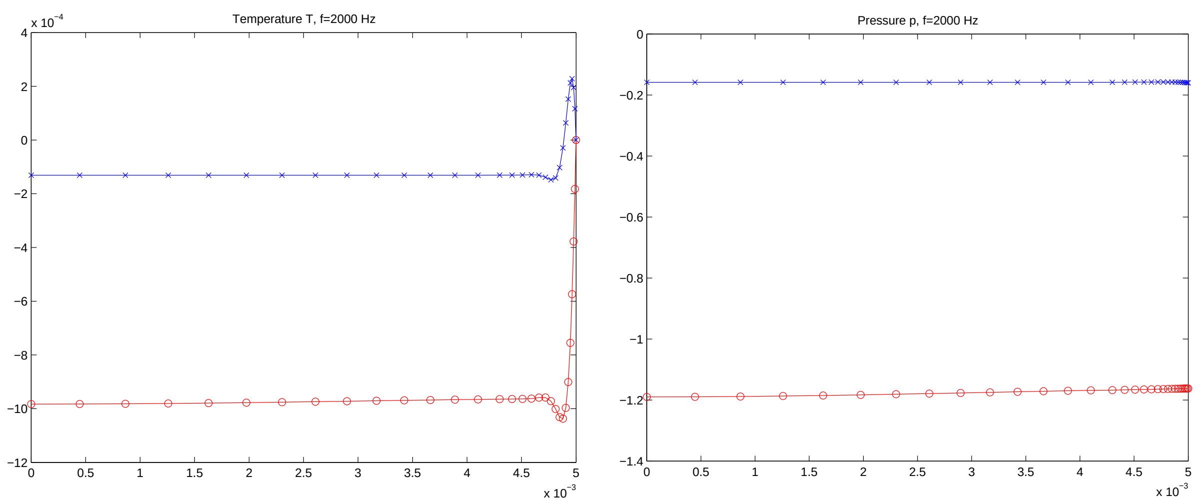

Figure 4: The real and imaginary parts of the temperature T and pressure p at the end $x_2 = L_3$ for $f = 2000$ Hz. The values of the real and imaginary parts at the nodes are displayed by $\times$ and $\circ$, respectively.

5 CONCLUDING REMARKS

The conclusions drawn from the order-of-magnitude analysis led us to formulating the system of acoustic field equations as a mixed problem. Among possible methods to discretizate the variational problem we have considered the method based on an enhanced MINI finite element. The reliability and applicability of the resulting finite element method have been demonstrated by numerical experiments.

It is noted that the performance of the method we have considered may degenerate if elements are highly stretched, see [6, p. 629] where this issue is discussed in the context of the Stokes equation. Owing to the need to model rapidly decaying boundary layer effects the use of such element shapes should not be ruled out, so it may be advantageous to explore the discretization of the mixed problem using other finite elements. Our current research also deals with the development of efficient preconditioners for iterative algorithms used in the solution of linear systems arising from the finite element discretization.

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