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FINITE ELEMENT SIMULATION OF THE ELECTRO-MECHANICAL PULL-IN PHENOMENON

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Abstract. *A fully convergent and efficient iteration scheme for determining the pull-in voltage of electro-mechanical systems is presented. The method separates the maximum displacement and mode of the displacement field. The mode is defined by the elasticity equations with the given electrostatic force. The maximum displacement is defined by an iterative pull-in equation based on lumped model electrostatics. The method may be applied to arbitrary geometries where the parameters for lumped electrostatics are computed by numerical differentiation. In planar geometries the parameters may be computed analytically which further enhances the method. The elasticity solver may be treated as a black box. The method is ideally suited, but not limited, to problems involving linear elasticity. The presented lumped model iteration scheme is implemented in Elmer finite element software. In the paper the method is demonstrated and verified in simple 2D cases. It is shown that with the present method the pull-in voltage may be defined roughly at the computational cost needed for solving one coupled electrostatic-elastic problem.*

1 INTRODUCTION

Pull-in phenomenon is a discontinuity related to the interplay of the elastic and electrostatic forces. When a potential difference is applied between a conducting structure and a ground level the structure deforms due to electrostatic forces. The elastic forces grow about linearly with displacement whereas the electrostatic forces grow inversely proportional to the square of the distance. When the voltage is increased the displacement grows until at some some point the growth rate of the electrostatic force exceeds that of the elastic force and the system cannot reach a force balance without a physical contact, thus pull-in occurs. The critical voltage is known as the pull-in voltage. The pull-in phenomenon is of great practical importance in the design of micro-electro-mechanical (MEMS) sensors and switches, for example.

The determination of the pull-in voltage and position requires the solution of a coupled electrostatic-elastic system [1]. Traditionally the pull-in analysis is done using voltage iteration (VI) which is a method of brute force. In the method the potential difference is gradually increased and for each value the coupled problem is solved iteratively. If a solution is obtained then the voltage is below the pull-in voltage otherwise the opposite is true. This scheme has no physical limitations but it is computationally very expensive. Around the pull-in position the convergence of the coupled problem may be slow. The accuracy of the scheme is determined by the step-size of the scanning. Economical and reasonably accurate scanning strategies usually require some initial estimate of the pull-in voltage.

The most simple pull-in geometry is the one-dimensional resonator for which an analytical expression for the pull-in position may be found. The formula is limited to cases which may be expressed by a lumped one-dimensional model. However, it turns out that it is possible to create a lumped model for the original distributed system. This lumped model can then be used to resolve the pull-in position and voltage on-the-fly. Even this strategy leads to an iteration scheme but in this case it converges accurately to the desired value. The method is not limited in generality and it may therefore be applied to cases of arbitrary geometries.

In the following we present the mathematical models for generalized and one dimensional cases. Thereafter we continue to develop the new scheme and finally evaluate it in a number of different cases.

2 MATHEMATICAL MODEL FOR ELECTRO-MECHANICAL SYSTEMS

For convenience we present the pull-in phenomenon in an idealized setting where the equations of electrostatics and elasticity obtain their simplest forms. This is, however, not a necessary limitation of the computational schemes used for the pull-in extraction. A schematic setup for the corresponding electro-mechanical system is shown in Figure 1.

The electric and magnetic fields are described by the Maxwell's equations. Assuming linear isotropic medium with constant permittivity, ε, and no free charges they result to

an equation for the electric potential, ϕ,

$$-\nabla \cdot \varepsilon \nabla \phi = 0 \quad \text{in } \Omega_e. \tag{1}$$

The electric field then yields

$$\boldsymbol{E} = -\nabla \phi. \tag{2}$$

At the conducting boundaries the electric potential is set. For simplicity we assume that there are only two possible values, one for the ground level and the other for the structure,

$$\begin{cases} \phi = 0 & \text{at } \Gamma_0, \\ \phi = V & \text{at } \Gamma_f. \end{cases} \tag{3}$$

The electric force acting on the surface of the structure is

$$\boldsymbol{f}_e = \boldsymbol{\tau} \cdot \boldsymbol{n} \quad \text{at } \Gamma_f, \tag{4}$$

where Maxwell's stress tensor is

$$\boldsymbol{\tau} = -\varepsilon \boldsymbol{E}\boldsymbol{E} + \frac{1}{2}\varepsilon E^2 \boldsymbol{I}. \tag{5}$$

The total energy of the electric field may be computed from the integral

$$\mathcal{E} = \int_{\Omega_e} \frac{1}{2}\varepsilon E^2 \, d\Omega. \tag{6}$$

From the energy also the capacitance of the system is obtained easily, $\mathcal{C} = 2\mathcal{E}/V^2$.

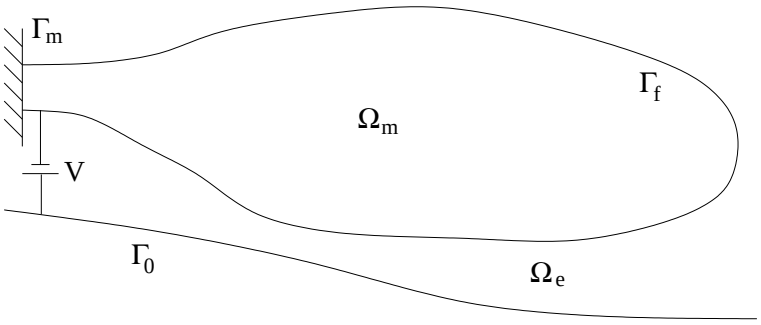

Figure 1: A schematic picture of the electro-mechanical system.

The electrostatic force causes the elastic structure to deform. Assuming linear elasticity and neglecting the body-forces the steady-state equation for elastic deformation may be written as

$$-\nabla \cdot \boldsymbol{\sigma} = 0 \quad \text{in } \Omega_m, \tag{7}$$

where the stress tensor for isotropic and isothermal materials may be expressed as

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon} + \lambda\nabla\cdot\boldsymbol{u}\boldsymbol{I}, \tag{8}$$

where $\boldsymbol{u}$ is the displacement field. μ and λ are the first and second Lame parameters respectively, and $\boldsymbol{\epsilon}$ is the strain tensor. Lame parameters in terms of Youngs modulus, Y, and Poisson ratio, κ, read

$$\mu = \frac{Y\kappa}{(1-\kappa)(1-2\kappa)} \quad \text{and} \quad \lambda = \frac{Y}{2(1+\kappa)}. \tag{9}$$

In steady-state analysis the structure must be fixed on part of the boundary which leads to Dirichlet conditions,

$$\boldsymbol{u} = 0 \quad \text{at } \Gamma_m. \tag{10}$$

At the conducting boundaries of the structure the boundary condition for the stress tensor yields

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{f}_e \quad \text{at } \Gamma_f. \tag{11}$$

This equation closes the system.

The determination of the pull-in voltage is equivalent with finding the largest value V for which the system has a solution. This is particularly difficult since a value exceeding the critical value easily leads to numerical divergence. If the computational implementation can handle the diverging solutions a bisection strategy for determining the pull in voltage may be used as in [1, 2]. Otherwise the solution must be approached from below which further cripples the VI scheme.

Note that as the elastic body deforms also the shape of the electrostatic domain changes. Therefore if the electrostatic equation is solved with typical volume discretization methods also the electrostatic domain must be made compatible with the deformed elastic structure. Typically this may be done by solving an additional deformation equation for the free space, now assuming that all the displacements on the boundaries are given. Such a procedure is often used also in conjunction with the modeling of fluid-structure interaction.

3 ONE-DIMENSIONAL PULL-IN ANALYSIS

In one-dimensional geometries the pull-in phenomenon is greatly simplified and the basic dependencies are easily visible. Now the displacement field is just a scalar u, and the mechanical force F_m and electric force F_e are algebraic functions of u. The pull-in position is determined by the equality of the forces and their derivatives,

$$\begin{cases} F_m &= F_e \\ \frac{\partial F_m}{\partial u} &= \frac{\partial F_e}{\partial u} \end{cases}. \tag{12}$$

Dividing the two equations we obtain

$$\frac{F_m}{K_m} = \frac{F_e}{K_e}, \tag{13}$$

where we have defined the electric and mechanical spring constants, $K_e = \frac{\partial F_e}{\partial u}$ and $K_m = \frac{\partial F_m}{\partial u}$, respectively. The equation is basically the same as the pull-in equation presented in [3] and it is valid for all voltage driven electro-mechanical systems. It may equally well be written as

$$u = \gamma \frac{F_e}{K_e}, \tag{14}$$

where $\gamma = K_m u / F_m$ describes the nonlinearity of the elasticity. For linear cases $\gamma = 1$ by definition. Equation (14) provides a starting point for the pull-in iteration scheme.

In the one-dimensional case the electric field has an analytical solution, $E = V/(d+u)$, where d is the initial aperture between the mass and the ground level. The electric energy now yields

$$\mathcal{E} = \frac{1}{2} \varepsilon A \frac{V^2}{d+u}, \tag{15}$$

where A is the area of the capacitor. The electric force and spring constant with fixed potential may be obtained by differentiation,

$$F_e = \frac{\partial \mathcal{E}}{\partial u} = -\frac{1}{2} \varepsilon A \frac{V^2}{(d+u)^2}, \quad \text{and} \quad K_e = \frac{\partial^2 \mathcal{E}}{\partial u^2} = \varepsilon A \frac{V^2}{(d+u)^3}. \tag{16}$$

Inserting the above formulas in Equation (14) and solving for the pull-in position gives $u = -d/3$. The relative displacement of $1/3$ is a characteristic value for this one-dimensional case. Also more complicated cases have values independent of the scale and of the material parameters.

The pull-in voltage depends also on the mechanical stiffness and may after some manipulation be obtained from

$$V_{PI} = \sqrt{\frac{8}{27} \frac{d^3 K_m}{\varepsilon A}}. \tag{17}$$

4 LUMPED MODEL ITERATION SCHEME FOR PULL-IN ANALYSIS

The displacements related to the solution of the electro-mechanical problem grow monotonically with the voltage. Therefore the whole displacement solution may uniquely be characterized by a single point in the solution. A natural choice is the point of maximum displacement in the structure. For this point it is in principle possible to make a one-dimensional lumped model that gives the same dependence between voltage, electric energy, and displacement as the original system of partial differential equations. From the lumped model the derivation of the pull-in voltage is much more simple than from the original system of equations.

Unfortunately, it is very difficult to make a lumped model for the whole operation range. Instead we generate the lumped model on-the-fly at the proposed pull-in position and use it to refine the position. When convergence is reached the lumped model is accurate at the pull-in position.

The displacement field may be written as

$$\boldsymbol{u} = U\frac{\boldsymbol{u}}{U} = U\tilde{\boldsymbol{u}}, \tag{18}$$

where U is the displacement of the lumped model and the field $\tilde{\boldsymbol{u}}$ represents only the shape of the displacement field. The division is the same as in the adaptive single mode (ASM) algorithm presented in [4]. However, we use a different strategy in defining U. Now, the electrostatic energy may be expressed as a function of these, $\mathcal{E} = \mathcal{E}(U, \tilde{\boldsymbol{u}})$.

Assuming that $\tilde{\boldsymbol{u}}$ is almost constant, we may compute expressions for the lumped electric force and spring constant,

$$F_e \approx \frac{\partial \mathcal{E}}{\partial U}, \quad \text{and} \quad K_e \approx \frac{\partial^2 \mathcal{E}}{\partial U^2}. \tag{19}$$

The success of the pull-in scheme depends on how well these lumped quantities are approximated. Basically there may be a small systematic error since $\tilde{\boldsymbol{u}}$ changes slightly as the force distribution varies depending on the maximum displacement. In some cases there are analytical approximations but otherwise numerical differentiation may have to be used in defining the parameters.

Because F_e and K_e are functions of U there must be an iteration scheme. The natural starting point is Equation (14) suggested by the one-dimensional case. The equation usually overestimates the changes and therefore relaxation is added to the scheme,

$$U^{(n+1)} = (1-\alpha)U^{(n)} + \alpha\gamma\frac{F_e}{K_e}. \tag{20}$$

For example, in the one-dimensional case a relaxation factor $\alpha = 2/3$ results to the correct value after just one iteration. In numerical tests on more complicated systems the relaxation always gave improved convergence. Also the elastic nonlinearity γ must be defined in a lumped manner as shown later.

When the estimate for the pull-in position U has been defined the computation of the pull-in voltage is straight-forward. It takes use of the facts that the elastic force is at least locally linear and the electric force is proportional to V^2. Therefore the new estimate for the pull-in voltage is

$$V^{(n+1)} = V^{(n)}\sqrt{\frac{U^{(n+1)}}{U^{(n)}}}. \tag{21}$$

After the new pull-in voltage is defined the electric field is scaled accordingly and used as a load to recompute the displacement field.

The pull-in position is determined by the electrostatic forces alone when the mode $\tilde{\boldsymbol{u}}$ is fixed. In order to make the iteration consistent the displacement field must be computed with the updated force field resulting from the pull-in position. Therefore there are two iteration levels: The inner iteration determines the pull-in position U and voltage V from the electrostatics, and the outer iteration determines the mode $\tilde{\boldsymbol{u}}$ corresponding to the updated electric field. In other words, the electrostatic equation is used to determine the amplitude of the displacement field and the elasticity equation only gives the shape of the displacement field. This way the discontinuity related to the numerical pull-in analysis is eliminated and the scheme converges to any desired accuracy.

4.1 Lumped model for arbitrary geometries

In generic cases the lumped electric force and spring constant must be computed numerically. The work done by the electric force is described by the integral

$$d\mathcal{E} = \int_\Gamma \boldsymbol{f}_e \cdot d\boldsymbol{u}\, dA = dU \int_\Gamma \boldsymbol{f}_e \cdot \tilde{\boldsymbol{u}}\, dA. \tag{22}$$

Therefore the lumped force becomes

$$F_e = \frac{\partial \mathcal{E}}{\partial U} = \int_\Gamma \boldsymbol{f}_e \cdot \tilde{\boldsymbol{u}}\, dA. \tag{23}$$

The lumped nature of the force is in seen in that the integral is filtered by the shape of displacement field. For example, constant forces must be multiplied with a factor $< \tilde{\boldsymbol{u}} >$ when they are interpreted in a lumped manner.

The spring coefficient may be computed from a differential of the lumped force,

$$K_e = \frac{F_e(U) - F_e((1-\beta)U)}{U\beta}, \tag{24}$$

where β is a small positive number. By using the backward differential the electric force obtained from the second field may also be used as a load for the elasticity equation. The computation of K_e therefore requires only one additional solution of Equation (1) with slightly modified geometry. It should be noted that using numerical differentiation requires that the lumped forces are computed in a very good accuracy. Otherwise numerical noise may ruin the scheme.

4.2 Lumped model for planar geometries

In planar geometries the creation of the lumped model may be greatly simplified. If the elastic structure is nearly aligned with the reference plane and aspect ratio of the aperture is large the electric field is effectively one-dimensional. The electric energy may then be obtained from

$$\mathcal{E} = \frac{1}{2}\varepsilon V^2 \int_A \frac{1}{d+u_n} dA, \tag{25}$$

where u_n is the normal component of the displacement field. In order to generate the lumped model we define $U = u_n|_{\max|u_n|}$ and $\tilde{u}_n = u_n/U$. Then the total energy of the electric field yields

$$\mathcal{E} = \frac{1}{2}\varepsilon V^2 \int_A \frac{1}{d + U\tilde{u}_n} dA, \tag{26}$$

and the lumped electric force and spring are obtained after differentiation,

$$F_e = \frac{\partial \mathcal{E}}{\partial U} = -\frac{1}{2}\varepsilon V^2 \int_A \frac{\tilde{u}_n}{(d + U\tilde{u}_n)^2} dA, \tag{27}$$

and

$$K_e = \frac{\partial^2 \mathcal{E}}{\partial U^2} = \varepsilon V^2 \int_A \frac{\tilde{u}_n^2}{(d + U\tilde{u}_n)^3} dA. \tag{28}$$

4.3 Lumped model for nonlinear elasticity

For nonlinear elasticity the pull-in position cannot be defined by the electrostatics alone. However, if the nonlinearity of the elasticity is small compared to the nonlinearity of the electrostatics the iteration scheme should still work. For generic cases the lumped nonlinearity of the elasticity must also be computed numerically.

After solving the elasticity equation with load $\boldsymbol{f}_e$ the lumped mechanical force F_m equals to the lumped electric force F_e. The lumped mechanical spring constant K_m may be computed by making a small variation in the electric load. The form of the load is assumed to be constant while the amplitude is modified. The numerical differentiation is otherwise analogous to Equation (24) and yields

$$\gamma = \frac{U}{F_m}\frac{\partial F_m}{\partial U} = \frac{\beta U(\boldsymbol{f}_e)}{U(\boldsymbol{f}_e) - U((1-\beta)\boldsymbol{f}_e)}. \tag{29}$$

Also here the backward differentiation ensures that only one additional solution for Equation (7) is required for determining γ. When convergence is obtained the nonlinearity is accurately described.

5 NUMERICAL RESULTS

The above presented scheme for performing the pull-in analysis has been implemented in Elmer software [5] that utilizes the finite element method (FEM) for the solution of partial differential equations. The scheme has been extensively used particularly for geometries where the electrostatic problem is accurately modeled by the one-dimensional approximation. Also some cases where numerical differentiation of lumped variables is required have been demonstrated. The structural problem has been modelled with linear 2D and 3D elasticity equations, and also with 2D plate equation. In the following we present some numerical results on different cases assuming linear elasticity.

5.1 Comparison of different schemes

In order to verify the consistency of the different schemes we compare them in a simple case. The geometry is a cantilever beam with length $l = 10$ μm, thickness $t = 1$ μm, and initial aperture $d = 1$ μm as shown in Figure 2. The material parameters are $Y = 169$ GPa and $\kappa = 0.32$.

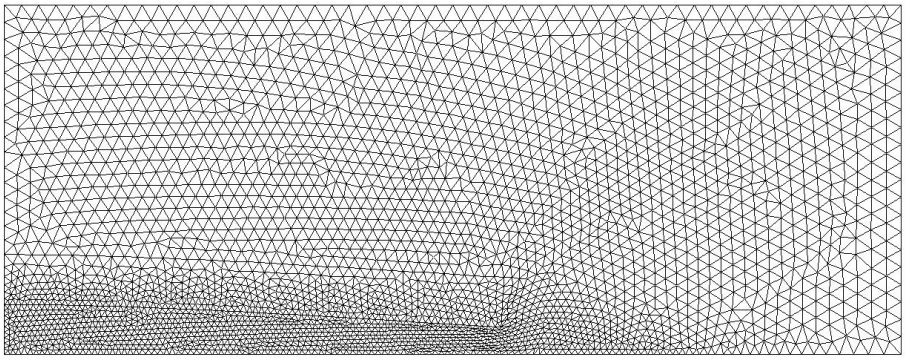

Figure 2: The computational mesh consisting of 6159 linear triangles at the computed pull-in position.

The voltage iteration (VI) scheme suggested that the pull-in voltage lies in between 773.8 and 773.9 V. The lower value gave a steady-state solution while the higher value resulted into pull-in. Typically around ten iterations were needed for testing one voltage. However, the number of iterations around the pull-in position grew to about a hundred. Without a priori knowledge of the pull-in voltage the total number of iterations in the VI scheme may therefore rise up to tens of thousands.

Scheme	β	V_{PI} (V)	$\mathcal{C}_{PI}$ (pF)	$\lvert U_{PI}/d \rvert$
VI	-	773.8	128.51	0.44960
LMI-ND	0.01	773.875	129.44	0.46058
LMI-ND	0.001	773.899	129.27	0.45852
LMI-ND	0.0001	773.892	129.25	0.45830
LMI-ND	0.00001	773.893	129.25	0.45828
LMI-AD	-	786.261	111.539	0.448084

Table 1: Comparison of different pull-in schemes for cantilever beam.

The lumped model iteration (LMI) scheme converged in around twenty iterations to four digit accuracy. The generic scheme with numerical differentiation (ND) was applied with different values of β. As may be seen in the Table 1 the results lie in the interval proposed by the VI scheme and are quite insensitive to the value of β. The magnitude of the electric field at the computed pull-in position is shown in Figure 3.

Also results using the analytical differentiation (AD) enabled by the one-dimensional electrostatics are presented for comparison. Some discrepancy is to be expected but the calculated pull-in voltage is still surprisingly close to the value of the generic model.

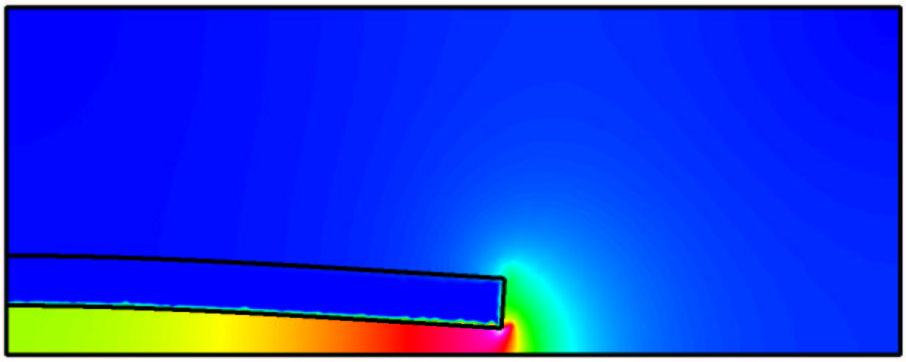

Figure 3: The magnitude of the electric field at the computed pull-in position.

The convergence of the lumped model iteration scheme was studied in more detail. In the computations the initial voltage was 100 V and the relaxation factor was $\alpha = 2/3$. The results are presented in Figure 4. The convergence is shown to be fast although not quite monotonic. The three lines represent cases with different number of sub-iterations where the pull-in position is corrected repeatedly without updating the displacement field. It may be seen that increasing the number from 1 to 2 slightly improves the convergence. However, a further increase to 3 does not make much difference. Therefore one or two sub-iterations seem to be optimal for overall performance of the scheme.

5.2 Comparison to literature

We also compared the results against some cases defined in [1]. These included different 2D structures; cantilever beams, fixed-fixed beams and an axisymmetric diaphragm. In all cases the Youngs modulus $Y = 169$ GPa and initial aperture $d = 1$ μm. The thickness t of the diaphragm was 20 μm and that of all other structures 3 μm. The length and the Poisson ratio varied from case to case. In these computations we used the analytical approach for the on-the-fly model lumping. The computational mesh was needed only for the structural domain and it consisted typically of around 20000 bilinear elements.

The results are presented in Table 2. As may be seen they are well in harmony with the reference values. The small deviations may be assumed to result from the discretization errors of the different strategies.

6 CONCLUSIONS

An lumped model iteration (LMI) method for computing the pull-in position and voltage was presented. The method separates the solution of the displacement mode and

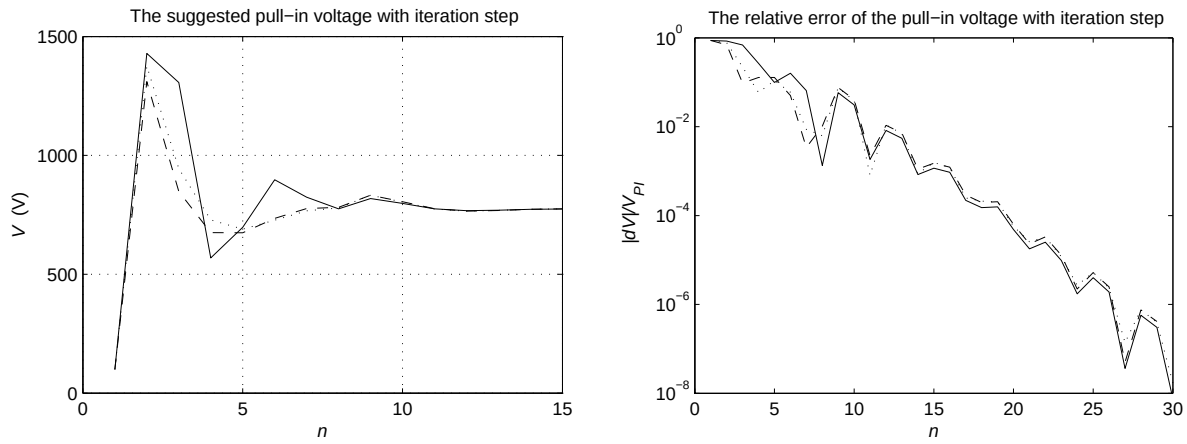

Figure 4: The convergence of the pull-in voltage with iteration step. The different lines represent cases with different number of sub-iterations for the pull-in position: solid line 1, dashed line 2 and dotted line 3.

geometry	V_{PI} (V)	$\lvert U_{PI}/d \rvert$	V_{PI}^1 (V)	V_{PI}^2 (V)
fixed-fixed beam, $l = 250$ μm, $\kappa = 0.06$	39.37	0.398	40.1	39.5
fixed-fixed beam, $l = 250$ μm, $\kappa = 0.32$	41.49	0.398	41.2	41.5
fixed-fixed beam, $l = 350$ μm, $\kappa = 0.06$	20.12	0.398	20.3	20.2
clamped beam, $l = 100$ μm, $\kappa = 0.06$	38.08	0.448	38.2	37.9
clamped beam, $l = 100$ μm, $\kappa = 0.32$	40.14	0.448	39.8	39.9
clamped beam, $l = 150$ μm, $\kappa = 0.06$	16.94	0.448	16.9	16.8
diaphragm, $r = 250$ μm, $\kappa = 0.30$	311.6	0.469	319	314

Table 2: Comparison of the results from the presented pull-in scheme to values in literature [1] where V_{PI}^1 refer to CoSolve simulations and V_{PI}^2 to closed form 2D model.

its amplitude. The amplitude is defined by an iterative lumped model scheme that is determined on-the-fly at the suggested pull-in position. When convergence is reached the model is exact at the pull-in position.

The method is ideally suited for cases where the electric spring constant is easily and accurately computed. Such is the case for planar geometries where the electrostatics equation reduces to a local one-dimensional problem. The scheme may also be applied to cases where the electric spring constant must be computed numerically. This is still much faster than the voltage iteration method, since only one additional electrostatic solution is required in defining the spring constant from the lumped forces. The iteration scheme has also the advantage that it converges firmly and rapidly towards the correct value.

The strength of the scheme compared to some earlier presented work is that it can use the generic solvers without any modifications. The applicability of the scheme is not

affected by the form or type of the elastic structure as the elasticity solver is used as a black box. The scheme may even be applied to cases where the elastic response to the electric forces is mildly nonlinear.

The iteration scheme was tested on a number of problems and the results were found to be in good agreement with those found in the literature. Convergence with four digits was obtained in about $10-20$ iteration cycles. The computational effort was therefore about the same as the one needed for solving a single coupled electrostatic-elastic problem.

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